

# Analysis of Algorithms, I

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Integer programming, set cover, approximation algorithms

- 1 Complexity classes
- 2 Integer Programming
- 3 Minimum-weight Set Cover
  - An integer programming formulation of Set Cover
  - The linear program relaxation
- 4 An approximation algorithm for Set Cover
  - Rounding the LP solution
  - An  $f$ -approximation algorithm for Set Cover

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## Definition 1.

We define  $\mathcal{P}$  to be the set of problems that can be solved by polynomial-time algorithms.

## Definition 2.

We define  $\mathcal{NP}$  to be the set of decision problems that have an efficient certifier.

## Fact 3.

$$\mathcal{P} \subseteq \mathcal{NP}$$

## Definition 4.

A problem  $X(D)$  is  $\mathcal{NP}$ -complete if

1.  $X(D) \in \mathcal{NP}$  and
2. for all  $Y \in \mathcal{NP}$ ,  $Y \leq_P X$ .

## Why we should care whether a problem is $\mathcal{NP}$ -complete

If a problem is  $\mathcal{NP}$ -complete, we need to *stop looking for efficient algorithms for the general problem.*

Instead we have a number of options, such as

1. **approximation algorithms**
  - ▶ mathematically rigorous basis to study heuristics
  - ▶ distinguish between various optimization problems in terms of how well they can be approximated
2. work on interesting special cases
3. study the average performance of the algorithm
4. *heuristics*

# The Traveling Salesman Problem (TSP)

**Tour:** a *simple* cycle that visits *every* vertex exactly once.

## Definition 5 (TSP(D)).

Given  $n$  cities  $\{1, \dots, n\}$ , a set of non-negative distances  $d_{ij}$  between every pair of cities and a budget  $B$ , is there a tour of length  $\leq B$ ?

Depending on the distances, TSP instances may be

- ▶ *Asymmetric:*  $d_{ij} \neq d_{ji}$
- ▶ *Symmetric:*  $d_{ij} = d_{ji}$
- ▶ *Metric:* satisfy the triangle inequality  $d_{ij} \leq d_{ik} + d_{kj}$
- ▶ *Euclidean:* e.g., cities are in  $\mathcal{R}^2$  hence city  $i$  corresponds to point  $(x_i, y_i)$ ; then  $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

# Similar problems with very different complexities

$\mathcal{NP}$ -complete	$\mathcal{P}$
max cut	min cut
longest path	shortest path
3D matching	matching
Hamiltonian cycle	Euler cycle
3-colorability	2-colorability
3-SAT	2-SAT
LCS of $n$ sequences	LCS of 2 sequences

More on  $\mathcal{NP}$ -completeness:

- ▶ *Computers and Intractability: A guide to the theory of  $\mathcal{NP}$ -completeness*, by Garey and Johnson
- ▶ *Computational Complexity*, by C. Papadimitriou

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# Integer Programming

**Integer programming (IP(D)):** Given a system of linear inequalities in  $n$  variables and  $m$  constraints with integer coefficients and an integer target value  $k$ , does it have an integer solution of value  $k$ ?

- ▶ Applications: production planning, scheduling trains, etc.

Example:

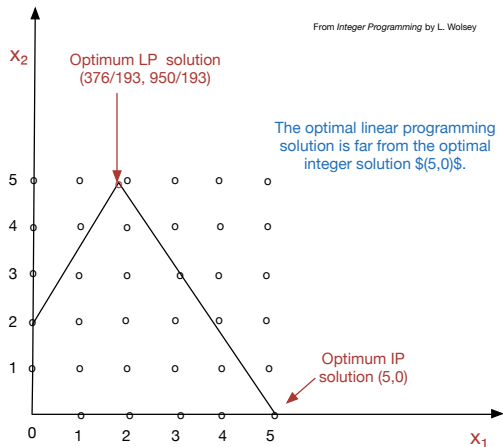
$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbf{Z}^n \end{aligned}$$

Here  $A$  is an  $m \times n$  matrix,  $\mathbf{b} \in \mathbf{R}^m$ ,  $\mathbf{c} \in \mathbf{R}^n$ ,  $\mathbf{x}$  is an integer vector with  $n$  components.

*What does the set of feasible solutions look like?*

# Rounding the LP is often insufficient

$$\begin{aligned} \max \quad & 1.00x_1 + 0.64x_2 \\ \text{subject to} \quad & x_1 \geq 0, x_2 \geq 0 \\ & 50x_1 + 31x_2 \leq 250 \\ & 3x_1 - 2x_2 \geq -4 \\ & x_1, x_2 \text{ integer} \end{aligned}$$



## *Is IP(D) hard?*

- ▶ IP(D) is in  $\mathcal{NP}$ .
- ▶ We can quickly solve LPs with several thousands of variables and constraints but there exist integer programs with 10 variables and 10 constraints that are very hard to solve.

## *Is IP(D) hard?*

- ▶ IP(D) is in  $\mathcal{NP}$ .
- ▶ We can quickly solve LPs with several thousands of variables and constraints but there exist integer programs with 10 variables and 10 constraints that are very hard to solve.
- ▶ This is not too surprising: integer programs restricted to solutions  $\mathbf{x} \in \{0, 1\}^n$  model **yes/no** decisions, which are generally hard.
- ▶ To formalize this intuition, we will reduce an  $\mathcal{NP}$ -complete problem to IP(D).

# Integer Programs for Vertex Cover and IS

First we formulate integer programs for two  $\mathcal{NP}$ -hard problems.

IP for Independent Set:

$$\begin{aligned} \max \quad & \sum_{i=0}^n x_i \\ \text{subject to} \quad & x_i + x_j \leq 1, \quad \text{for every } (i, j) \in E \\ & x_i \in \{0, 1\}, \quad \text{for every } i \in V \end{aligned}$$

IP for Vertex Cover:

$$\begin{aligned} \min \quad & \sum_{i=0}^n x_i \\ \text{subject to} \quad & x_i + x_j \geq 1, \quad \text{for every } (i, j) \in E \\ & x_i \in \{0, 1\}, \quad \text{for every } i \in V \end{aligned}$$

## Claim 1.

$$\text{VC(D)} \leq_P \text{IP(D)}$$

## Proof.

Reduction from arbitrary instance  $(G = (V, E), k)$  of  $\text{VC(D)}$  to the following integer program with target value  $k$ :

$$\begin{aligned} \min \quad & 0 \\ \text{subject to} \quad & x_i + x_j \geq 1, \quad \text{for every } (i, j) \in E \\ & \sum_{i=1}^n x_i = k \\ & x_i \in \{0, 1\}, \quad \text{for every } i \in V \end{aligned}$$

Equivalence of the instances is straightforward. □

# Similar problems with very different complexities (*new*)

$\mathcal{NP}$ -complete	$\mathcal{P}$
max cut	min cut
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3-colorability	2-colorability
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LCS of $n$ sequences	LCS of 2 sequences
integer programming	linear programming

The theory of integer and linear programming and duality can guide the design of approximation algorithms, and exact solutions, for hard problems.

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# Minimum-weight Set Cover

## Input

- ▶ a set  $E = \{e_1, e_2, \dots, e_n\}$  of  $n$  elements
- ▶ a collection of subsets of these elements  $S_1, S_2, \dots, S_m$ , where each  $S_j \subseteq E$
- ▶ a non-negative weight  $w_j$  for every subset  $S_j$

## Output

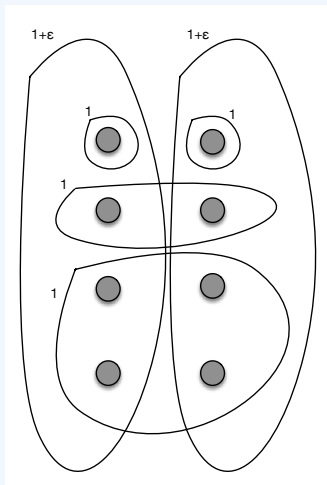
A minimum-weight collection of subsets that cover all of  $E$ .

In symbols: find an  $I \subseteq \{1, \dots, m\}$  such that  $\bigcup_{i \in I} S_i = E$  and

$\sum_{i \in I} w_i$  is minimum.

(Unweighted Set Cover:  $w_j = 1$  for all  $j$ )

# Example instance of Set Cover



$n = 8$  ground elements,  $m = 6$  subsets with weights  
 $w_1 = w_2 = w_3 = w_4 = 1$ ,  $w_5 = w_6 = 1 + \epsilon$ .

# Motivation: detect computer viruses with IBM AntiVirus

*Motivation:* detect salient features of boot sector viruses that do not occur in typical applications

- ▶ **Ground elements:** known boot sector viruses ( $n \approx 150$ )
- ▶ **Sets:** labelled by some three-byte sequence occurring in these viruses but not occurring in typical computer applications ( $m \approx 21000$ ); each set consisted of all the viruses that contained the three-byte sequence
- ▶ **Objective:** output a small number of such sequences (much smaller than 150) that *cover* all known viruses

# Reduction via generalization

## Claim 2.

Set-Cover(D) is  $\mathcal{NP}$ -complete.

## Proof.

Reduction from VC(D). Input instance:  $(G = (V, E), k)$ .

- ▶ Set  $E = \{e_1, \dots, e_m\}$  to be the set of ground elements we want to *cover*.
- ▶ For every vertex  $j$ , set  $S_j$  to be the set of edges (ground elements) that are incident to –hence *covered* by– vertex  $j$ .
- ▶ Set  $w_j = 1$  for all  $1 \leq j \leq n$ .

Equivalence of instances: input graph has a vertex cover of size  $k$  if and only if  $E$  has a set cover of weight  $k$ . □

# Designing the integer program for Set Cover

**Variables:** we introduce one variable per set  $S_j$ ; intuitively,

$$x_j = \begin{cases} 1, & \text{if } S_j \text{ is included in the solution} \\ 0, & \text{otherwise} \end{cases}$$

**Constraints:** ensure that every element is *covered*:

for every element  $e_i$ , at least one of the sets  $S_j$   
containing  $e_i$  appears in the final solution

**Objective function:** minimize the sum of the weights of the sets included in the solution

# An integer programming formulation of Set Cover

Integer program for **Set Cover**:

$$\begin{aligned} \min \quad & \sum_{j=1}^m w_j x_j \\ \text{subject to} \quad & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \\ & x_j \in \{0, 1\}, \quad \text{for every } 1 \leq j \leq m \end{aligned}$$

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Integer program for **Set Cover**:

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Let  $Z_{IP}^*$  be the optimum value of this integer program;  
 $OPT$  be the value of the optimum solution to **Set Cover**.

$$Z_{IP}^* = OPT.$$

△ We cannot solve this integer program efficiently (*why?*).

# LP relaxation: a bound for the value of the IP

LP relaxation for **Set Cover**:

$$\begin{array}{ll} \min_{\mathbf{x} \geq \mathbf{0}} & \sum_{j=1}^m w_j x_j \\ \text{subject to} & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{array}$$



# LP relaxation: a bound for the value of the IP

LP relaxation for **Set Cover**:

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- ▶ Every feasible solution to the original IP is a feasible solution to the LP relaxation.
- ▶ The value of any feasible solution to the original IP is the same in the LP (the objectives are the same).
- ▶ Let  $Z_{LP}^*$  be the optimum value of the LP relaxation.

$$Z_{LP}^* \leq Z_{IP}^* = OPT$$

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# Rounding the solution to the LP

LP relaxation for **Set Cover**:

$$\begin{aligned} \min_{\mathbf{x} \geq \mathbf{0}} \quad & \sum_{j=1}^n w_j x_j \\ \text{subject to} \quad & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{aligned}$$

- ▶ Let  $x^*$  be an optimal solution to the LP relaxation.
- ▶ Let  $f_i = \#$  subsets  $S_j$  where element  $e_i$  appears.
- ▶ Let  $f = \max_{1 \leq i \leq n} f_i$ .
- ▶ Set

$$\hat{x}_j = \begin{cases} 1, & \text{if } x_j^* \geq 1/f \\ 0, & \text{if } x_j^* < 1/f \end{cases}$$

# Rounding yields a feasible solution to the original IP

The collection of sets  $S_j$  with  $\hat{x}_j = 1$  cover all the elements.

- ▶ Consider the optimal solution  $x^*$  for the LP relaxation.
- ▶ Fix any element  $e_i$ ; recall that  $e_i$  appears in  $f_i$  subsets.
- ▶ For simplicity, relabel these subsets as  $S_1, S_2, \dots, S_{f_i}$ . Then the optimal solution satisfies the constraint

$$x_1^* + x_2^* + \dots + x_{f_i}^* \geq 1$$

Let  $x_m^*$  be the maximum of  $x_1^*, x_2^*, \dots, x_{f_i}^*$ . Then

$$x_m^* \geq \frac{1}{f_i} \geq \frac{1}{f}$$

- ⇒ Our rounding procedure guarantees that, for every element  $e_i$ , at least one set  $S_j$  that *covers*  $e_i$  is chosen.

# An $f$ -approximation algorithm for Set Cover

How far is the solution obtained by the rounding procedure above from to the *optimal* solution to Set Cover?

- ▶ We do **not** know *OPT*!
- ▶ **But** we have a **bound** for it: the value  $Z_{LP}^*$  of the LP relaxation!

Recall that we set  $\hat{x}_j = 1$  if and only if  $x_j^* \geq 1/f$ . Then

$$\begin{aligned}\sum_j w_j \hat{x}_j &\leq \sum_j w_j (f x_j^*) = f \sum_j w_j x_j^* \\ &= f \cdot Z_{LP}^* \leq f \cdot OPT\end{aligned}$$

## Definition 6.

An  $\alpha$ -approximation algorithm for an optimization problem is a polynomial-time algorithm that, for all instances of the problem, produces a solution whose value is within a factor of  $\alpha$  of the value of the optimal solution.

## Remark 1.

- ▶  $\alpha$  is the approximation ratio or approximation factor
- ▶ For *minimization* problems,  $\alpha > 1$ .
- ▶ For *maximization* problems,  $\alpha < 1$ .

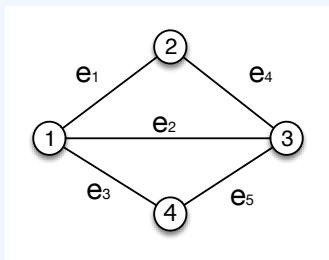
**Example 1:** the rounding procedure described on slide 53 gives an  $f$ -approximation algorithm for **Set Cover**:

- ▶ it can be completed in polynomial-time
- ▶ it always returns a solution whose value is at most  $f$  times the value of the optimal solution.

**Remark:** if an element appears in too many sets (e.g.,  $f = \Omega(n)$ ), this algorithm does not provide a good approximation guarantee.

**Example 2:** a 2-approximation algorithm for VC is a polynomial-time algorithm that always returns a solution whose value is at most twice the value of the optimal solution.

## A 2-approximation algorithm for $VC$



- ▶ Let  $E = \{e_1, \dots, e_m\}$  be the set of edges in the graph.
- ▶ Let  $S_j$  be the set of edges (ground elements) that are covered by vertex  $j$ .
- ▶ For every edge  $e_i$ ,  $f_i = 2$ :  $e_i$  appears in exactly two subsets (*why?*).
- ▶ Hence  $f = \max_{1 \leq i \leq m} f_i = 2$ .