Analysis of Algorithms, I
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Eleni Drinea
Computer Science Department

Columbia University

Minimum spanning trees: Prim's and Kruskal's algorithms

Outline

1. Minimum Spanning Trees (MSTs)
   - Prim's algorithm
   - Kruskal's algorithm
   - More MST algorithms

Today

1. Minimum Spanning Trees (MSTs)
   - Prim's algorithm
   - Kruskal's algorithm
   - More MST algorithms

The problem

Motivation: build the cheapest communication network over a set of locations.

Input: a weighted, undirected graph $G = (V, E, w)$

Output: a subset of edges $E_T \subseteq E$ such that

1. the graph $T = (V, E_T)$ is connected;
2. $\sum_{e \in E_T} w(e)$ is minimal.

Minimum weight Spanning Trees (MST)

Remark 1.

The graph $T = (V, E_T)$ is a tree: if there is a cycle, remove any edge from the cycle and obtain a connected graph with less cost.

Definition 1 (Spanning tree of a graph $G = (V, E)$).

A tree that spans all the nodes in V.

Output (restated): a minimum weight spanning tree of G.

Remarks

- Brute-force won't work: even simple graphs have many spanning trees—*how many in a simple cycle?*
- #spanning trees in the complete graph on n vertices: n^{n-2}

The cut property

Definition 2 (Cut).

A cut $(S, V - S)$ is a bipartition of the vertices.

Claim 1 (Cut property).

Assume all edge weights are distinct. Let $S \subset V$ $(S \neq \emptyset)$. Let e be the minimum-weight edge with one endpoint in S and the other in $V - S$. Then every MST contains e.

Remark 2.

The assumption of distinct edge weights is just for the purposes of the analysis; we will show how to remove it later.

Proof of the cut property

Notation: $w(T) = \sum_{e \in E_T} w(e)$

We will derive a contradiction by using an exchange argument.

- Let T' be a minimum-weight spanning tree that does not contain $e = (u, v)$.
- Then there must be some other path P in T' from u to v.
- Starting at u, follow the vertices of P: since (u, v) crosses from S to $V - S$, there must be some first vertex $v' \in V - S$ on P. Let u' be the last vertex before it in S.
- Then $e' = (u', v') \in E_{T'}$ **and** e' crosses between S, $V - S$.

Proof of the cut property (cont'd)

Exchange e with e' to obtain the set of edges

$$E_T = E_{T'} + \{e\} - \{e'\}.$$

T is a spanning tree:

- T is connected: any path in T' that used $e' = (u', v')$ is rerouted to follow P from u' to u, (u, v) and P from v to v'.
- T is acyclic (*why?*).

Since both e' and e cross between S and $V - S$ but e is the lightest edge with this property, $w(e) < w(e')$. Thus

$$w(T) < w(T').$$

Using the cut property to design MST algorithms

The cut property says: construct MST greedily by taking the lightest edge across two regions not yet connected.

`Generic-MST`$(G = (V, E, w))$

$E_T = \emptyset$ // the set of edges that will form our MST

while $|E_T| \leq n - 1$ **do**

Pick $S \subseteq V$ s.t. no edge in E_T crosses between S, $V - S$

Let $e \in E$ be a lightest edge that crosses between S, $V - S$

$E_T = E_T \cup \{e\}$

end while

Prim's algorithm

In Prim's algorithm, the edges in E_T always form a subtree which is a partial MST and S is chosen to be the set of this subtree's vertices.

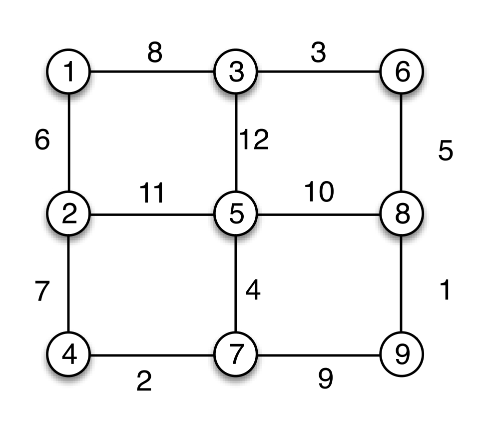
In other words:

1. Start with a root node s.
2. **Greedily** grow a tree outward from s by adding the node that can be attached as cheaply as possible at every step.

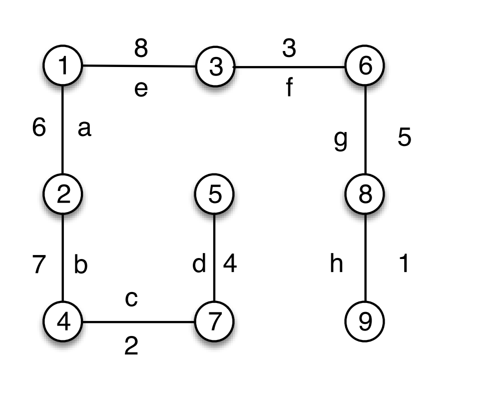
Detailed description of Prim's algorithm

1. $E_T = \emptyset$
2. Maintain a set $S \subseteq V$ on which a spanning tree has been constructed so far. Initially, $S = \{s\}$.
3. In each iteration, update
   3.1 $S = S \cup \{v\}$, where v is the vertex in $V - S$ that minimizes the attachment cost:
   $$\min_{\substack{u \in S \\ (u,v) \in E}} w_{uv}.$$
   3.2 $E_T = E_T \cup \{e\}$

Example graph



Prim's MST for example graph (letters indicate the order in which edges were added)



Prim's: correctness

Follows directly from the Cut property.

Let S be the set of vertices on which a partial MST has been constructed.
At every iteration an edge (u, v) is added such that

- $u \in S$, $v \in V - S$;
- (u, v) is the lightest edge that crosses between S and $V - S$.

Implementing Prim's algorithm

Similarly to Dijkstra's algorithm,

- store every node $v \in V - S$ in a priority queue Q, e.g., implemented as a binary min-heap (key= weight of the lightest edge between some node in S and v). Initially, $S = \{s\}$.
- maintain two arrays
  - $dist[v]$: stores the weight of the lightest edge between v and any vertex in S (in Dijkstra, it stored a conservative overestimate of the distance of v from the source s)
  - $prev[v]$: stores the node responsible for adding v to S

Pseudocode: how does this compute $T = (V, E_T)$?

```
Prim(G = (V, E, w), s)
    for u ∈ V do
        dist[v] = ∞; prev[v] = NIL
    end for
    dist[s] = 0
    Q = {V; dist}
    S = ∅
    while Q ≠ ∅ do
        u = ExtractMin(Q)
        S = S ∪ {u}
        for (u, v) ∈ E and v ∈ V − S do
            if dist[v] > w(u, v) then
                dist[v] = w(u, v)
                prev[v] = u
                DecreaseKey(Q, v)
            end if
        end for
    end while
```

Further implementations of Prim's algorithm

Notation: $|V| = n$, $|E| = m$

Implementation	`ExtractMin`	`Insert/ DecreaseKey`	Time
Array	$O(n)$	$O(1)$	$O(n^2)$
Binary heap	$O(\log n)$	$O(\log n)$	$O((n+m)\log n)$
d-ary heap	$O(d \log n)$	$O(\log n)$	$O((nd+m)\frac{\log n}{\log d})$
Fibonacci heap	$O(\log n)$	$O(1)$ amortized	$O(n \log n + m)$

- Optimal choice for $d \approx m/n$ (the *average* degree of the graph)
- d-ary heap works well for both sparse and dense graphs
  - *If $m = n^{1+x}$, what is the running time of Prim's algorithm using a d-ary heap?*
- **Amortized** analysis: *coming up in the next lecture*

Kruskal's algorithm

Short description: at every step, add to E_T the *lightest* edge that does not create a cycle with the edges already in E_T.

Thus, at all times, E_T is a subset of an MST.

Alternative view: merging partial trees

Initially, every vertex forms its own trivial tree (no edges). Maintain a *forest* of trees at all times.

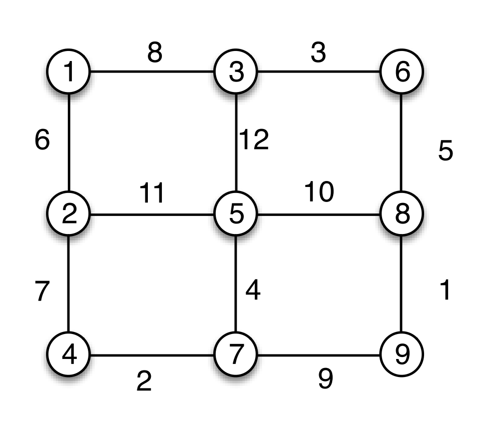
Let $T(v)$ be the tree where vertex v belongs.

1. Initialize $E_T = \emptyset$
2. `Sort` the edges by increasing weight.
3. For every edge $e = (u, v)$ in **increasing order** of weight:
    - If u and v belong to the same tree, discard e.
    - Else
        - $E_T = E_T \cup \{e\}$;
        - merge $T(u)$, $T(v)$ into a single tree.

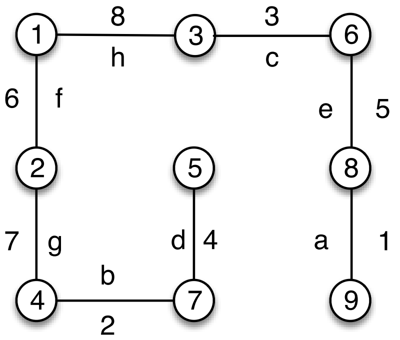
△ *Need a data structure that allows*

1. *to check if u, v belong to the same tree;*
2. *for updates to reflect the merging of two trees into one.*

Example graph



Kruskal's MST for example graph (letters indicate the order in which edges were added)



Correctness

- Let (u, v) be the edge added at the current iteration.
- Let S be the set of nodes that have a path to u by edges in A just before (u, v) is added; then $u \in S$ but $v \notin S$.
- Also, (u, v) must be the first edge between S and $V - S$ encountered so far: otherwise, if such an edge was encountered before, it would have been added to A since its inclusion would not cause a cycle.

$\Rightarrow$ (u, v) is the lightest edge that crosses between S and $V - S$

- By the Cut Property, (u, v) belongs to the MST.

Implementing Kruskal's algorithm

Kruskal's algorithm maintains a forest of trees at all times, starting from n trivial trees (no edges).

Want a data structure that maintains a **collection of disjoint sets** and supports operations:

1. `MakeSet`(u): Given an element u, create a new tree containing only u. Target worst-case time: $O(1)$
2. `Find`(u): Given an element u, find which tree u belongs to. Target worst-case time: $O(\log n)$
3. `Union`(u, v): Merge the tree containing u and the tree containing v into a single tree. Target worst-case time: $O(\log n)$

Pseudocode

```
Kruskal(G = (V, E, w))
    E_T = ∅
    Sort(E) by w
    for u ∈ V do MakeSet(u)
    end for
    for (u, v) ∈ E by increasing w do
        if Find(u) ≠ Find(v) then
            E_T = E_T ∪ {(u, v)}
            Union(u, v)
        end if
    end for
```

Running time analysis

- Sorting: $O(m \log m) = O(m \log n)$
- n `Makeset`() operations: $O(n)$
- $2m$ `Find`() operations: $2m \cdot O(\log n)$
- $\leq n - 1$ `Union`() operations: $n \cdot O(\log n)$

Running time: $O(m \log n)$

When is it safe to not include an edge to the MST?

Fact 3 (The Cycle Property).

Assume that all edge costs are distinct. Let C be any cycle in G, and let edge (u, v) be the heaviest edge in C. Then e does not belong to any MST of G.

Proof of the cycle property

- Let T be a spanning tree that contains e. We want to show that T is not optimal.
- To this end, we will exchange e for some e' to get a spanning tree T' with less weight.
- First, delete e from T; T is now partitioned into two components: the set S containing u and the set $V - S$ containing v.

$\Rightarrow$ We want an edge e' with one endpoint in S and another in $V - S$ so as to reconnect them.

Proof of the cycle property (cont'd)

- We can find such an edge by following the cycle C.
- Consider the edges of C except for e: they form a path from u to v.
- So if we start at u, following this path, at some point there is an edge e' that crosses from S to $V - S$. Construct

$$E_{T'} = E_T - \{e\} + \{e'\}.$$

- Now T' is connected and has $n - 1$ edges. Moreover, since e is the heaviest edge in the cycle

$$w(T') < w(T).$$

More MST algorithms

Fact 3 yields yet another algorithm for finding an MST.

`Reverse-Delete`$(G = (V, E, w))$

- Start with the full graph.
- Sort the edges in decreasing weight.
- Repeatedly delete edges in order of decreasing weight, so long as the graph does not become disconnected.

More MST algorithms: combine the Cut property (to add edges) and the Cycle property (to eliminate edges).

△ Such algorithms may be subtle to implement.

Removing the assumption of unequal edge weights

- Suppose some edges have equal weights.
- Slightly perturb all edge weights by different, **tiny** amounts.

$\Rightarrow$ All edge weights are now distinct.

- Apply the algorithms discussed in the previous sections.

Remark 3.

Perturbations serve as tie-breakers: edges whose weights differed before still have the same relative order.