Analysis of Algorithms, CSOR W4231
CSOR W4231

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Computer Science Department

Columbia University

Data compression and huffman coding
1. Data compression

2. Symbol codes and optimal lossless compression

3. Prefix codes

4. Prefix codes and trees

5. The Huffman algorithm
1. Data compression

2. Symbol codes and optimal lossless compression

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4. Prefix codes and trees

5. The Huffman algorithm
Data compression: find compact representations of data

Data compression standards

- jpeg for image transmission
- mp3 for audio content, mpeg2 for video transmission
- utilities: gzip, bzip2

All of the above use the Huffman algorithm as a basic building block.
Data representation

- An organism’s genome consists of *chromosomes* (giant linear DNA molecules)
- *Chromosome maps*: sequences of hundreds of millions of bases (symbols from \{A, C, G, T\}).
- **Goal**: store a chromosome map with 200 million bases.

*How do we represent a chromosome map?*
An organism’s genome consists of *chromosomes* (giant linear DNA molecules).

*Chromosome maps*: sequences of hundreds of millions of bases (symbols from \{A, C, G, T\}).

**Goal**: store a chromosome map with 200 million bases.

*How do we represent a chromosome map?*

- **Encode** every symbol that appears in the sequence separately by a fixed length binary string.
- **Codeword** \( c(x) \) for symbol \( x \): a binary string encoding \( x \) of length \( \ell(x) \)
Example code

- Alphabet $\mathcal{A} = \{A, C, G, T\}$ with 4 symbols
- Encode each symbol with 2 bits

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</tr>
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<td>01</td>
</tr>
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Output of encoding: the concatenation of the codewords for every symbol in the input sequence
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Output of encoding: the concatenation of the codewords for every symbol in the input sequence

Example

- Input sequence: $ACGTA$
Example code

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- Encode each symbol with 2 bits

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Output of encoding: the concatenation of the codewords for every symbol in the input sequence

Example

- Input sequence: $ACGTA$A
- Output: $c(A)c(C)c(G)c(T)c(A)c(A) = 000110110000$
- Total length of encoding $= 6 \cdot 2 = 12$ bits.
1. Data compression

2. Symbol codes and optimal lossless compression

3. Prefix codes

4. Prefix codes and trees

5. The Huffman algorithm
Symbol code: a set of codewords where every input symbol is encoded separately.
Symbol codes

Symbol code: a set of codewords where every input symbol is encoded separately.

Examples of symbol codes

- $C_0 = \{00, 01, 10, 11\}$ is a symbol code for $\{A, C, G, T\}$.
- ASCII encoding system: every character and special symbol on the computer keyboard is encoded by a different 7-bit binary string.
Symbol code: a set of codewords where every input symbol is encoded \textit{separately}.

Examples of symbol codes

- $C_0 = \{00, 01, 10, 11\}$ is a symbol code for \{A, C, G, T\}.
- ASCII encoding system: every character and special symbol on the computer keyboard is encoded by a different 7-bit binary string.

\textbf{Remark 1.}

$C_0$ and ASCII are \textit{fixed-length} symbol codes: each codeword has the same length.
Decoding $C_0$?
Unique decodability

Decoding $C_0$

- read 2 bits of the output;
- print the symbol corresponding to this codeword;
- continue with the next 2 bits.
Unique decodability

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- read 2 bits of the output;
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Remark 2.

- This decoding algorithm works for ASCII (replace 2 by 7)
- $C_0$, ASCII: distinct input sequences have distinct encodings
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Remark 2.

- *This decoding algorithm works for ASCII (replace 2 by 7)*
- $C_0$, ASCII: *distinct input sequences have distinct encodings*

Definition 1.

A symbol code is *uniquely decodable* if, for any two distinct input sequences, their encodings are distinct.
Lossless compression: compress and decompress without errors.

- Uniquely decodable codes allow for lossless compression.
- A symbol code achieves optimal lossless compression when it produces an encoding of minimum length for its input (among all uniquely decodable symbol codes).
- Huffman algorithm: provides a symbol code that achieves optimal lossless compression.
Chromosome map consists of 200 million bases as follows:

<table>
<thead>
<tr>
<th>alphabet symbol $x$</th>
<th>frequency $freq(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>110 million</td>
</tr>
<tr>
<td>$C$</td>
<td>5 million</td>
</tr>
<tr>
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- \( A \) appears much more often than the other symbols.
- It might be best to encode \( A \) with fewer bits.
- Unlikely that the fixed-length encoding \( C_0 \) is optimal.
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3. Prefix codes
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## Variable-length encodings

<table>
<thead>
<tr>
<th>Code $C_1$</th>
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<th>codeword $c(x)$</th>
</tr>
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<tbody>
<tr>
<td>$A$</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$C$</td>
<td></td>
<td>00</td>
</tr>
<tr>
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<td></td>
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</tr>
<tr>
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Prefix codes

Variable-length encodings

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<td>$T$</td>
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$C_1$ is not unique decodable! E.g., 101110: *how to decode it?*
Variable-length encodings

<table>
<thead>
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<td>$C$</td>
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- $C_2$ is uniquely decodable.
- $C_2$ is such that no codeword is a prefix of another.
Prefix codes

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- $C_2$ is uniquely decodable.
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Definition 2 (prefix codes).

A symbol code is a prefix code if no codeword is a prefix of another.
1. Scan the binary string from left to right until you’ve seen enough bits to match a codeword;

2. Output the symbol corresponding to this codeword.
   - Since no other codeword is a prefix of this codeword or contains it as a prefix, this sequence of bits cannot be used to encode any other symbol.

3. Continue starting from the next bit of the bit string.
Decoding prefix codes

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Thus prefix codes allow for

- unique decoding;
- fast decoding (the end of a codeword is instantly recognizable).
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Thus prefix codes allow for
   ▶ unique decoding;
   ▶ fast decoding (the end of a codeword is instantly recognizable).

Examples of prefix codes: $C_0, C_2$
Decoding a prefix code is very fast.

Would like to focus on prefix codes (rather than all uniquely decodable symbol codes) for achieving optimal lossless compression.

Information theory guarantees this: for every uniquely decodeable code, exists a prefix code with the same codeword lengths.

So we can solely focus on prefix codes for optimal compression.
Chromosome map: *do we gain anything by using $C_2$ instead of $C_0$ when compressing the map of 200 million bases?*

<table>
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<tr>
<th>Input</th>
<th>Code $C_0$</th>
<th>Code $C_2$</th>
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</thead>
<tbody>
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<td>symbol $x$</td>
<td>$freq(x)$</td>
<td>$x$</td>
</tr>
<tr>
<td>$A$</td>
<td>110 million</td>
<td>$A$</td>
</tr>
<tr>
<td>$C$</td>
<td>5 million</td>
<td>$C$</td>
</tr>
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<td></td>
<td></td>
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<td>$c(x)$</td>
</tr>
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<td>110 million</td>
<td>$A$</td>
<td>00</td>
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- $C_0$: 2 bits $\times 200$ million symbols $= 400$ million bits
- $C_2$: $1 \cdot 110 + 3 \cdot 5 + 3 \cdot 25 + 2 \cdot 60 = 320$ million bits
- Improvement of 20% in this example
The optimal prefix code problem

Input:
- Alphabet $\mathcal{A} = \{a_1, \ldots, a_n\}$
- Set $P = \{p_1, \ldots, p_n\}$ of empirical probabilities over $\mathcal{A}$ such that $p_i = \text{Pr}[a_i]$

Output: a binary prefix code $C^* = \{c(a_1), c(a_2), \ldots, c(a_n)\}$ for $(\mathcal{A}, P)$, where codeword $c(a_i)$ has length $\ell_i$ and is such that its expected length

$$L(C^*) = \sum_{a_i \in \mathcal{A}} p_i \cdot \ell_i$$

is minimum among all binary prefix codes.
Example

Chromosome example

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<th>Input</th>
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<tr>
<td>$A$</td>
<td>110/200</td>
<td>$A$ 00</td>
<td>$A$ 0</td>
</tr>
<tr>
<td>$C$</td>
<td>5/200</td>
<td>$C$ 01</td>
<td>$C$ 110</td>
</tr>
<tr>
<td>$G$</td>
<td>25/200</td>
<td>$G$ 10</td>
<td>$G$ 111</td>
</tr>
<tr>
<td>$T$</td>
<td>60/200</td>
<td>$T$ 11</td>
<td>$T$ 10</td>
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- $L(C_0) = 2$
- $L(C_2) = 1.6$
- *Coming up*: $C_2$ is the output of the Huffman algorithm, hence an optimal encoding for $(\mathcal{A}, P)$. 
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A binary tree $T$ is a rooted tree such that each node that is not a leaf has at most two children.

Binary tree for a prefix code: a branch to the left represents a 0 in the encoding and a branch to the right a 1.

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Binary tree for prefix code $C_2$
Properties of binary trees representing prefix codes

1. Where do alphabet symbols appear in the tree?
2. What do codewords correspond to in the tree?
3. Consider the tree corresponding to the optimal prefix code. Can it have internal nodes with one child?
Properties of binary trees representing prefix codes

1. Symbols must appear at the leaves of the tree $T$ (why?)
   $\Rightarrow$ $T$ has $n$ leaves.

2. Codewords $c(a_i)$ are given by root-to-leaf paths.

   Recall that $\ell_i$ is the length of the codeword $c(a_i)$ for input symbol $a_i$. Therefore, on the tree $T$, $\ell_i$ corresponds to the depth of $a_i$ (we assume that the root is at depth 0).

   $\Rightarrow$ Can rewrite the expected length of the prefix code as:

   $$L(C) = \sum_{a_i \in A} p_i \cdot \ell_i = \sum_{1 \leq i \leq n} p_i \cdot \text{depth}_T(a_i) = L(T).$$

3. Optimal tree must be full: all internal nodes must have exactly two children (why?).
Claim 1.

There is an optimal prefix code, with corresponding tree $T^*$, in which the two lowest frequency characters are assigned to leaves that are siblings in $T^*$ at maximum depth.
More on optimal tree

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There is an optimal prefix code, with corresponding tree $T^*$, in which the two lowest frequency characters are assigned to leaves that are siblings in $T^*$ at maximum depth.

Proof.

By an exchange argument: start with a tree for an optimal prefix code and transform it into $T^*$. 

\Halmos
Proof of Claim 1

- Let $T$ be the tree for the optimal prefix code.
- Let $\alpha, \beta$ be the two symbols with the smallest probabilities, that is, $\Pr[\alpha] \leq \Pr[\beta] \leq \Pr[s]$ for all $s \in A - \{\alpha, \beta\}$.
- Let $x$ and $y$ be the two siblings at maximum depth in $T$.

We want $L(T) \geq L(T')$.
Proof of Claim 1

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- Let $x$ and $y$ be the two siblings at maximum depth in $T$.

We want $L(T') \geq L(T'')$
How do the expected lengths of the two trees compare?

\[
L(T) - L(T') = \sum_{a_i \in A} \Pr[a_i] \cdot \text{depth}_T(a_i) - \sum_{a_i \in A} \Pr[a_i] \cdot \text{depth}_{T'}(a_i)
\]

\[
= \Pr[\alpha] \cdot \text{depth}_T(\alpha) + \Pr[x] \cdot \text{depth}_T(x)
- \Pr[\alpha] \cdot \text{depth}_{T'}(\alpha) - \Pr[x] \cdot \text{depth}_{T'}(x)
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- \Pr[\alpha] \cdot \text{depth}_T(x) - \Pr[x] \cdot \text{depth}_T(\alpha)
\]

\[
= (\Pr[\alpha] - \Pr[x]) \cdot (\text{depth}_T(\alpha) - \text{depth}_T(x)) \geq 0
\]

- The third equality follows from the exchange.
- Similarly, exchanging \( \beta \) and \( y \) in \( T' \) yields \( L(T') - L(T'') \geq 0 \).
- Hence \( L(T) - L(T'') \geq 0 \).
- Since \( T \) is optimal, it must be \( L(T) = L(T'') \).
- So \( T'' \) is also optimal.

The claim follows by setting \( T^* \) to be \( T'' \).
Claim 1 tells us how to build the optimal tree greedily!

1. Find the two symbols with the lowest probabilities.
2. Remove them from the alphabet and replace them with a new meta-character with probability equal to the sum of their probabilities.
   - **Idea:** this meta-character will be the parent of the two deleted symbols in the tree.
3. Recursively construct the optimal tree using this process.

**Greedy algorithms:** make a local (myopic) decision at every step that optimizes some criterion and eventually show that this is the optimal way for building the entire solution.
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Huffman algorithm

\textbf{Huffman}(\mathcal{A}, P)

\textbf{if} \quad |\mathcal{A}| = 2 \textbf{then}
\begin{itemize}
\item Encode one symbol using 0 and the other using 1
\end{itemize}
\textbf{end if}

Let \(\alpha\) and \(\beta\) be the two symbols with the lowest probabilities
Let \(\nu\) be a new meta-character with probability \(\Pr[\alpha] + \Pr[\beta]\)
Let \(\mathcal{A}_1 = \mathcal{A} - \{\alpha, \beta\} + \{\nu\}\)
Let \(P_1\) be the new set of probabilities over \(\mathcal{A}_1\)
\(T_1 = \text{Huffman}(\mathcal{A}_1, P_1)\)
\textbf{return} \(T\) as follows: replace leaf node \(\nu\) in \(T_1\) by an internal node, and add two children labelled \(\alpha\) and \(\beta\) below \(\nu\).

\textbf{Remark 3.}

\textit{Output of Huffman procedure is a binary tree} \(T\); \textit{the code for} \((\mathcal{A}, P)\) \textit{is its corresponding prefix code.}
Example: recursive Huffman for chromosome map

Recursive call 1: \texttt{Huffman}({A, C, G, T}, \{\frac{110}{200}, \frac{5}{200}, \frac{25}{200}, \frac{60}{200}\})

Recursive call 2: \texttt{Huffman}({A, \nu CG, T}, \{\frac{110}{200}, \frac{30}{200}, \frac{60}{200}\})

Recursive call 3: \texttt{Huffman}({A, \nu CG T}, \{\frac{110}{200}, \frac{90}{200}\})
Correctness

**Proof:** by induction on the size of the alphabet $n \geq 2$.

- **Base case.** For $n = 2$, Huffman is optimal.

- **Hypothesis.** Assume that Huffman returns the optimal prefix code for alphabets of $n$ symbols.

- **Induction Step.** Let $\mathcal{A}$ be an alphabet of size $n + 1$, $P$ the corresponding set of probabilities.

  Let $T_1$ be the optimal (by the hypothesis) tree returned by our algorithm for $(\mathcal{A}_1, P_1)$, where $\mathcal{A}_1, P_1, T_1$ as in the pseudocode. Let $T$ be the final tree returned for $(\mathcal{A}, P)$ by our algorithm. We claim that $T$ is optimal.

  We will prove the claim by contradiction. Assume $T^*$ is the optimal tree for $(\mathcal{A}, P)$ such that

  \[ L(T^*) < L(T). \]  

  (1)
Fact 3.

Let $T$ be a binary tree representing a prefix code. If we replace sibling leaves $\alpha, \beta$ in $T$ by a meta-character $\nu$ where $\Pr[\nu] = \Pr[\alpha] + \Pr[\beta]$, we obtain a tree $T_1$ such that

$$L(T) = L(T_1) + (\Pr[\alpha] + \Pr[\beta]).$$

Proof.

Notation: $d_T(a_i) = \text{depth}_T(a_i)$

• $\alpha, \beta$ are sibling leaves in $T$, hence $d_T(\alpha) = d_T(\beta)$.
• $T$ differs from $T_1$ only in that $\alpha, \beta$ are replaced by $\nu$. Since $d_{T_1}(\nu) = d_T(\alpha) - 1$, we obtain

$$L(T) - L(T_1) = \Pr[\alpha]d_T(\alpha) + \Pr[\beta]d_T(\beta) - (\Pr[\alpha] + \Pr[\beta])d_{T_1}(\nu)$$

$$= \Pr[\alpha] + \Pr[\beta].$$

(2)
Claim 1 guarantees there is such an optimal tree for \((A, P)\) where \(\alpha, \beta\) appear as siblings at maximum depth.

W.l.o.g. assume that \(T^*\) is such an optimal tree. By Fact 3, if we replace siblings \(\alpha, \beta\) in \(T^*\) by \(\nu'\) where \(\Pr[\nu'] = \Pr[\alpha] + \Pr[\beta]\), the resulting tree \(T_1^*\) satisfies \(L(T^*) = L(T_1^*) + (\Pr[\alpha] + \Pr[\beta])\).

Similarly, the tree \(T\) returned by the Huffman algorithm satisfies \(L(T) = L(T_1) + (\Pr[\alpha] + \Pr[\beta])\).

By the induction hypothesis, we have \(L(T_1^*) \geq L(T_1)\) since \(T_1\) is optimal for alphabets of size \(n\). Hence

\[
L(T^*) = L(T_1^*) + \Pr[\alpha] + \Pr[\beta] \geq L(T_1) + \Pr[\alpha] + \Pr[\beta] = L(T),
\]

where the inequality follows from the induction hypothesis.

Equation (3) contradicts Assumption (1). Thus \(T\) must be optimal.
Implementation and running time

1. Straightforward implementation: $O(n^2)$ time
2. Store the alphabet symbols in a min priority queue implemented as a binary min-heap with keys their probabilities
   - **Operations**: Initialize ($O(n)$), Extract-min ($O(\log n)$), Insert ($O(\log n)$)
   
   Total time: $O(n \log n)$ time

For an iterative implementation of Huffman, see your textbook.
Example: iterative Huffman for chromosome map

<table>
<thead>
<tr>
<th>symbol</th>
<th>Pr(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>110/200</td>
</tr>
<tr>
<td>C</td>
<td>5/200</td>
</tr>
<tr>
<td>G</td>
<td>25/200</td>
</tr>
<tr>
<td>T</td>
<td>60/200</td>
</tr>
</tbody>
</table>

→

<table>
<thead>
<tr>
<th>symbol</th>
<th>c(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>110</td>
</tr>
<tr>
<td>G</td>
<td>111</td>
</tr>
<tr>
<td>T</td>
<td>10</td>
</tr>
</tbody>
</table>

**Step 1**

```
0 1
5/200 25/200
C G
```

**Step 2**

```
0 1
60/200 30/200
T G C
```

**Step 3**

```
0 1
110/200 90/200
A T C G
```
Beyond Huffman coding

- Huffman algorithm provides an optimal symbol code.
- Codes that encode larger blocks of input symbols might achieve better compression.
- Storage on noisy media: what if a bit of the output of the compressor is flipped?
  - Decompression cannot carry through.
  - Need error correction on top of compression.