**CSOR W4231.002** **Notes on the Master Theorem** (rev: Jan 2017)

Often when analyzing a divide & conquer algorithm, we obtain a recurrence for its running time of the following form

$$T(n) \quad = \quad aT(\frac{n}{b}) + cn^k \tag{1}$$

In words, on input size $n$, the algorithm generates $a$ subproblems, each of size $n/b$; combining these subproblems to obtain the overall solution requires time polynomial in $n$, specifically $cn^k$.

Such recurrences appear frequently so it is useful to know asymptotic bounds for them in terms of $a, b$ and $k$ (as we will see, $c$ does not affect the asymptotic solution). To this end, we will analyze the recursion tree for this recurrence (see Figure 1).

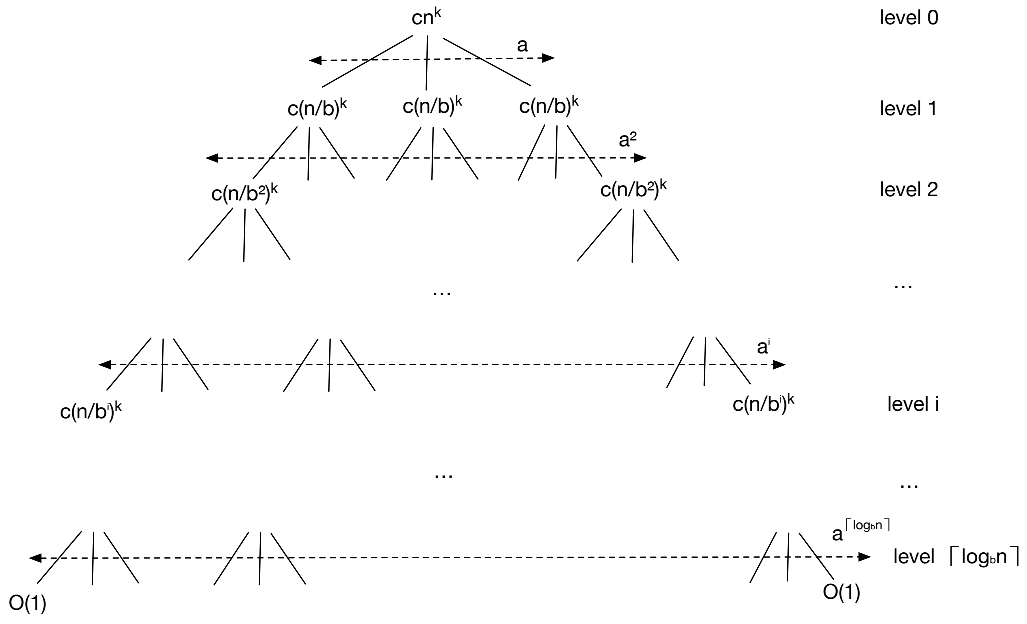

Figure 1: The recursion tree for recurrence (1). $a$ is the branching factor, $b$ is the factor by which the input size shrinks at every recursive call and $cn^k$ is the time required to combine the solutions to the subproblems into the overall solution for input size is $n$. The smallest possible size of a subproblem is $O(1)$; typically, solving input instances of small constant size requires constant time $c$.

Note that

- $a$ is the branching factor of the tree: every subproblem gives rise to $a$ new subproblems at the next level of the tree; thus
  1. at level 1, we have $a$ subproblems
  2. at level 2, **each** of the $a$ subproblems in level 1 gives rise to $a$ new subproblems; therefore there are a total of $a^2$ subproblems
  3. at level 3, **each** of the $a^2$ subproblems in level 2 generates $a$ new subproblems; therefore there are a total of $a^3$ subproblems
  4. at the level $i$, there are $a^i$ subproblems

- $b$ is the factor by which the input size shrinks at every level; thus

  1. at level 1, the input size of each subproblem shrinks by a factor of $b$, that is, from $n$ it now becomes $n/b$;
  2. at level 2, the input size of each subproblem further shrinks by a factor of $b$, that is, from $n/b$ it now becomes $(n/b)/b = n/b^2$;
  3. at level 3, the input size of each subproblem again shrinks by a factor of $b$, hence becomes $(n/b^2)/b = n/b^3$;
  4. at level $i$, the size of each subproblem is $n/b^i$

$\Rightarrow$ at level $i$, the amount of work spent on each subproblem of size $n/b^i$ is [1]:

$$c\left(\frac{n}{b^i}\right)^k$$

$\Rightarrow$ at level $i$, the work spent on **all** subproblems is

$$a^i c\left(\frac{n}{b^i}\right)^k = cn^k\left(\frac{a}{b^k}\right)^i$$

We need one more observation before we can compute the total work spent on the recursion tree.

**Fact 1** *The depth of the tree in Figure 1 is $\lceil \log_b n \rceil$ levels.*

**Proof.** The last level of the recursion tree, call it $d$, consists of subproblems of size 1. Since at level $i$ subproblems have size $n/b^i$, we are looking for $d$ such that

$$\frac{n}{b^d} = 1 \Rightarrow d = \log_b n$$

Since $d$ is an integer, $d = \lceil \log_b n \rceil$. □

We are now ready to derive a bound for $T(n)$ by computing the total work spent on this recursion tree, which is given by the sum of the work spent at each level of the tree:

$$T(n) \quad = \quad \sum_{i=0}^{\lceil \log_b n \rceil} cn^k\left(\frac{a}{b^k}\right)^i = cn^k \sum_{i=0}^{\lceil \log_b n \rceil}\left(\frac{a}{b^k}\right)^i \tag{2}$$

Note that $T(n)$ depends on a sum over $\lceil \log_b n \rceil$ terms of a geometric progression with common ratio $a/b^k$ and initial value $(a/b^k)^0 = 1$. Depending on the value of the common ratio $a/b^k$, this sum will exhibit the following behavior:

1. $\frac{a}{b^k} = 1$; in this case, we have

$$\sum_{i=0}^{\lceil \log_b n \rceil}\left(\frac{a}{b^k}\right)^i = \sum_{i=0}^{\lceil \log_b n \rceil} 1 = \lceil \log_b n \rceil + 1 = \Theta(\log_b n) \tag{3}$$

2. $\frac{a}{b^k} < 1$; in this case, you can show that the sum of the entire geometric progression is dominated by its initial value, that is,

$$\sum_{i=0}^{\lceil \log_b n \rceil}\left(\frac{a}{b^k}\right)^i = \Theta\left(\left(\frac{a}{b^k}\right)^0\right) = \Theta(1) \tag{4}$$

[1]Recall that the amount of work spent on combining the subproblems when the the input size is $n$ is $cn^k$.

3. $\frac{a}{b^k} > 1$; again, you can show that the sum of the entire geometric progression is now dominated by its last term, that is,

$$\sum_{i=0}^{\lceil \log_b n \rceil} \left(\frac{a}{b^k}\right)^i = \Theta\Big(\left(\frac{a}{b^k}\right)^{\log_b n}\Big) = \Theta\Big(\left(\frac{a^{\log_b n}}{b^{k \log_b n}}\right)\Big) = \Theta\Big(\frac{n^{\log_b a}}{n^k}\Big) \tag{5}$$

Plugging back equations (3), (4), (5) into equation (2), we summarize our findings in the following theorem.

**Theorem 1 (Master theorem)** *If $T(n) = aT(\lceil n/b \rceil) + O(n^k)$ for some constants $a > 0$, $b > 1$, $k \geq 0$, then*

$$T(n) = \begin{cases} O(n^{\log_b a}) & , \text{ if } a > b^k \\ O(n^k \log n) & , \text{ if } a = b^k \\ O(n^k) & , \text{ if } a < b^k \end{cases}$$