

Analysis of Algorithms, I

CSOR W4231

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Representative NP-complete problems: TSP, Set Cover

- 1 Review of last lecture
- 2 Representative \mathcal{NP} -complete problems
- 3 Integer Programming
- 4 Minimum-weight Set Cover
 - An integer programming formulation of Set Cover
 - The linear program relaxation
- 5 An approximation algorithm for Set Cover
 - Rounding the LP solution
 - An f -approximation algorithm for Set Cover

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Definition 1.

We define \mathcal{P} to be the set of problems that can be solved by polynomial-time algorithms.

Definition 2.

We define \mathcal{NP} to be the set of decision problems that have an efficient certifier.

Fact 3.

$$\mathcal{P} \subseteq \mathcal{NP}$$

Definition 4.

A problem $X(D)$ is \mathcal{NP} -complete if

1. $X(D) \in \mathcal{NP}$ and
2. for all $Y \in \mathcal{NP}$, $Y \leq_P X$.

How do we show that a problem is \mathcal{NP} -complete?

Suppose we had an \mathcal{NP} -complete problem X .

To show that another problem Y is \mathcal{NP} -complete, we use **transitivity of reductions**. So we “only” need show that

1. $Y \in \mathcal{NP}$
2. $X \leq_P Y$

The first \mathcal{NP} -complete problem

Theorem 5 (Cook-Levin).

Circuit SAT is \mathcal{NP} -complete.

Satisfiability of boolean functions

SAT: Given a formula ϕ in CNF with n variables and m clauses, is ϕ satisfiable?

3SAT: Given a formula ϕ in CNF with n variables and m clauses such that each clause has exactly 3 literals, is ϕ satisfiable?

Circuit-SAT: Given a boolean combinatorial circuit C , is there an assignment of truth values to its inputs that causes the output to evaluate to 1?

Lemma 6.

Circuit-SAT \leq_P *SAT*, *SAT* \leq_P *3SAT* and *3SAT* \leq_P *IS(D)*

Common pitfalls when showing \mathcal{NP} -completeness

1. Carry out the reduction in the wrong direction
2. Reduce from a problem not known to be \mathcal{NP} -complete
3. Exponential-time transformations
 - ▶ Subsets, permutations
4. Neglect to carefully prove both directions of equivalence of the original and the derived instances; that is, x is a **yes** instance of X *if and only if* $y = R(x)$ is a **yes** instance of Y
5. Neglect to show that the problem is in \mathcal{NP}

Suggestions

- ▶ You should think carefully which problem is most suitable to reduce from
- ▶ In absence of other ideas, reduce from 3SAT

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The Traveling Salesman Problem (TSP)

Tour: a *simple* cycle that visits *every* vertex exactly once.

Definition 7 (TSP(D)).

Given n cities $\{1, \dots, n\}$, a set of non-negative distances d_{ij} between every pair of cities and a budget B , is there a tour of length $\leq B$?

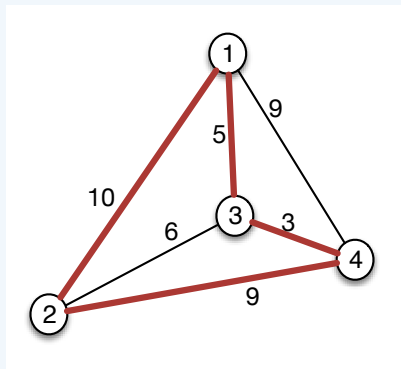
Equivalently, is there a **permutation** π such that

1. $\pi(1) = \pi(n+1) = 1$; that is, we start and end at city 1
2. the total distance travelled satisfies

$$\sum_{i=1}^n d_{\pi(i)\pi(i+1)} \leq B$$

Application: Google street view car

Example instance of TSP



Depending on the distances, TSP instances may be

- ▶ *Asymmetric*: $d_{ij} \neq d_{ji}$
- ▶ *Symmetric*: $d_{ij} = d_{ji}$
- ▶ *Metric*: satisfy the triangle inequality $d_{ij} \leq d_{ik} + d_{kj}$
- ▶ *Euclidean*: e.g., cities are in \mathcal{R}^2 hence city i corresponds to point (x_i, y_i) ; then $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

A related problem and hardness of TSP(D)

Hamiltonian Cycle: Given a graph $G = (V, E)$, is there a simple cycle that visits every vertex exactly once?

Claim 1.

Hamiltonian Cycle *is* \mathcal{NP} -complete.

Proof: Reduction from 3SAT (e.g., see your textbook).

Claim 2.

Symmetric TSP(D) *is* \mathcal{NP} -complete.

Proof: reduction from undirected Hamiltonian Cycle.

Proof of Claim 2 (Hamiltonian Cycle \leq_P TSP(D))

1. Start from an arbitrary instance of undirected **Hamiltonian Cycle**, that is, an undirected graph $G = (V, E)$.
2. Construct the following instance $(G' = (V', E', w), B)$ of TSP(D): G' is a *complete* weighted graph with $V' = V$ such that for every edge $e \in E'$,

$$w_e = \begin{cases} 1, & \text{if } e \in E \\ 2, & \text{otherwise} \end{cases}$$

3. Set the budget $B = n$.

This completes the reduction transformation.

Equivalence of the instances is straightforward:

- ▶ If G has a hamiltonian cycle, that cycle is a tour of length n in G' .
- ▶ If G' has a tour of length n , it must consist of edges of weight 1 (*why?*); thus all these edges appear in G .

Concluding remarks on TSP

- ▶ Claim 1 also holds for directed Hamiltonian cycle. An exact analog of the proof of Claim 2 then shows that asymmetric TSP is \mathcal{NP} -complete.
- ▶ It is possible to reduce Hamiltonian cycle to Euclidean TSP, thus showing that even Euclidean TSP is \mathcal{NP} -complete.
- ▶ However, these problems are not similar in terms of how well they can be approximated: it is possible to provide very good approximate solutions to Euclidean TSP, which is not the case for Symmetric TSP.

Packing and partitioning problems

- ▶ **Set Packing:** given a set U of a elements, a collection S_1, S_2, \dots, S_b of subsets of U , and a number k , is there a collection of at least k subsets such that no two of them intersect?
- ▶ **3D-Matching:** Given disjoint sets B, G, H , each of size n , and a set of triples $T \subseteq B \times G \times H$, is there a set of n triples in T , no two of which have an element in common?
Reduction from 3SAT.

Numerical problems

- ▶ **Subset sum:** Given natural numbers w_1, \dots, w_n and a (large) target weight W , is there a subset of w_1, \dots, w_n that adds up exactly to W ?

Applications: cryptography, scheduling

- ▶ **Minimum-weight solution to linear equations:** Given a system of linear equations in n variables with integer constants, and an integer $B \leq n$, does it have a rational solution with at most B non-zero entries?

Applications: coding theory, signal processing

Similar problems with very different complexities

\mathcal{NP} -complete	\mathcal{P}
max cut	min cut
longest path	shortest path
3D matching	matching
Hamiltonian cycle	Euler cycle
3-colorability	2-colorability
3-SAT	2-SAT
LCS of n sequences	LCS of 2 sequences

More on \mathcal{NP} -completeness:

- ▶ *Computers and Intractability: A guide to the theory of \mathcal{NP} -completeness*, by Garey and Johnson
- ▶ *Computational Complexity*, by C. Papadimitriou

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Integer Programming

Integer programming (IP(D)): Given a system of linear inequalities in n variables and m constraints with integer coefficients and an integer target value k , does it have an integer solution of value k ?

- ▶ Applications: production planning, scheduling trains, etc.

Example:

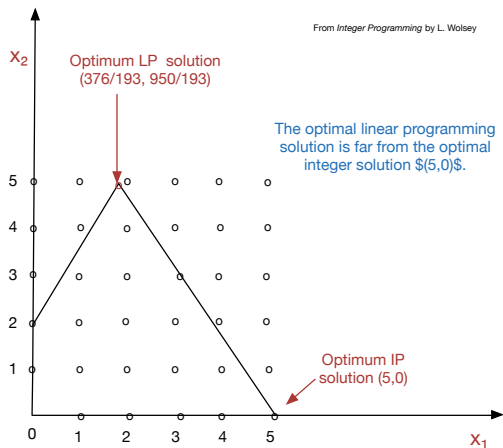
$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbf{Z}^n \end{aligned}$$

Here A is an $m \times n$ matrix, $\mathbf{b} \in \mathbf{R}^m$, $\mathbf{c} \in \mathbf{R}^n$, \mathbf{x} is an integer vector with n components.

What does the set of feasible solutions look like?

Rounding the LP is often insufficient

$$\begin{aligned} \max \quad & 1.00x_1 + 0.64x_2 \\ \text{subject to} \quad & x_1 \geq 0, x_2 \geq 0 \\ & 50x_1 + 31x_2 \leq 250 \\ & 3x_1 - 2x_2 \geq -4 \\ & x_1, x_2 \text{ integer} \end{aligned}$$



Is IP(D) hard?

- ▶ $\text{IP}(\mathcal{D})$ is in \mathcal{NP} .
- ▶ We can quickly solve LPs with several thousands of variables and constraints but there exist integer programs with 10 variables and 10 constraints that are very hard to solve.

Is IP(D) hard?

- ▶ IP(D) is in \mathcal{NP} .
- ▶ We can quickly solve LPs with several thousands of variables and constraints but there exist integer programs with 10 variables and 10 constraints that are very hard to solve.
- ▶ This is not too surprising: integer programs restricted to solutions $\mathbf{x} \in \{0, 1\}^n$ model **yes/no** decisions, which are generally hard.
- ▶ To formalize this intuition, we will reduce an \mathcal{NP} -complete problem to IP(D).

Integer Programs for Vertex Cover and IS

First we formulate integer programs for two \mathcal{NP} -hard problems.

IP for Independent Set:

$$\begin{aligned} \max \quad & \sum_{i=0}^n x_i \\ \text{subject to} \quad & x_i + x_j \leq 1, \quad \text{for every } (i, j) \in E \\ & x_i \in \{0, 1\}, \quad \text{for every } i \in V \end{aligned}$$

IP for Vertex Cover:

$$\begin{aligned} \min \quad & \sum_{i=0}^n x_i \\ \text{subject to} \quad & x_i + x_j \geq 1, \quad \text{for every } (i, j) \in E \\ & x_i \in \{0, 1\}, \quad \text{for every } i \in V \end{aligned}$$

Claim 3.

$$\text{VC(D)} \leq_P \text{IP(D)}$$

Proof.

Reduction from arbitrary instance $(G = (V, E), k)$ of VC(D) to the following integer program with target value k :

$$\begin{aligned} \min \quad & 0 \\ \text{subject to} \quad & x_i + x_j \geq 1, \quad \text{for every } (i, j) \in E \\ & \sum_{i=1}^n x_i \leq k \\ & x_i \in \{0, 1\}, \quad \text{for every } i \in V \end{aligned}$$

Equivalence of the instances is straightforward. □

Similar problems with very different complexities (*new*)

\mathcal{NP} -complete	\mathcal{P}
max cut	min cut
longest path	shortest path
3D matching	matching
Hamiltonian cycle	Euler cycle
3-colorability	2-colorability
3-SAT	2-SAT
LCS of n sequences	LCS of 2 sequences
integer programming	linear programming

The theory of integer and linear programming and duality can guide the design of approximation algorithms, and exact solutions, for hard problems.

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Minimum-weight Set Cover

Input

- ▶ a set $E = \{e_1, e_2, \dots, e_n\}$ of n elements
- ▶ a collection of subsets of these elements S_1, S_2, \dots, S_m , where each $S_j \subseteq E$
- ▶ a non-negative weight w_j for every subset S_j

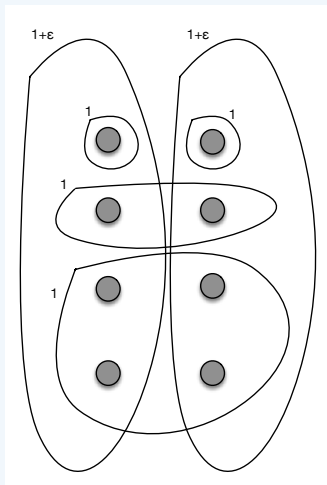
Output

A minimum-weight collection of subsets that cover all of E .

In symbols: find an $I \subseteq \{1, \dots, m\}$ such that $\cup_{i \in I} S_i = E$ and $\sum_{i \in I} w_i$ is minimum.

(Unweighted Set Cover: $w_j = 1$ for all j)

Example instance of Set Cover



$n = 8$ ground elements, $m = 6$ subsets with weights
 $w_1 = w_2 = w_3 = w_4 = 1$, $w_5 = w_6 = 1 + \epsilon$.

Motivation: detect computer viruses

Motivation (IBM AntiVirus): detect features of boot sector viruses that do not occur in typical applications; then use them to discover more viruses

- ▶ **Ground elements:** known boot sector viruses ($n \approx 150$)
 - ▶ **Sets:** labelled by some three-byte sequence occurring in these viruses but not occurring in typical computer applications ($m \approx 21000$); each set consisted of all the viruses that contained the three-byte sequence
 - ▶ **Output:** a small number of such sequences—much smaller than 150—that *cover* all known viruses
- ⇒ use the small set cover as features in a *neural classifier* to determine presence of a boot sector virus
- ⇒ *detect* new viruses (many boot sector viruses are written by modifying existing ones)

Reduction via generalization

Claim 4.

Set-Cover(D) is \mathcal{NP} -complete.

Proof.

Reduction from VC(D). Input instance: $(G = (V, E), k)$.

- ▶ Set $E = \{e_1, \dots, e_m\}$ to be the set of ground elements we want to *cover*.
- ▶ For every vertex j , set S_j to be the set of edges (ground elements) that are incident to—hence *covered* by—vertex j .
- ▶ Set $w_j = 1$ for all $1 \leq j \leq n$.

Equivalence of instances: input graph has a vertex cover of size k if and only if E has a set cover of weight k . □

Forming the integer program for Set Cover

Variables: we introduce one variable per set S_j ; intuitively,

$$x_j = \begin{cases} 1, & \text{if } S_j \text{ is included in the solution} \\ 0, & \text{otherwise} \end{cases}$$

Constraints: ensure that every element is *covered*:

for every element e_i , at least one of the sets S_j
containing e_i appears in the final solution

Objective function: minimize the sum of the weights of the sets included in the solution

An integer programming formulation of Set Cover

Integer program for **Set Cover**:

$$\begin{aligned} \min \quad & \sum_{j=1}^m w_j x_j \\ \text{subject to} \quad & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \\ & x_j \in \{0, 1\}, \quad \text{for every } 1 \leq j \leq m \end{aligned}$$

An integer programming formulation of Set Cover

Integer program for **Set Cover**:

$$\begin{aligned} \min \quad & \sum_{j=1}^m w_j x_j \\ \text{subject to} \quad & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \\ & x_j \in \{0, 1\}, \quad \text{for every } 1 \leq j \leq m \end{aligned}$$

Let Z_{IP}^* be the optimum value of this integer program;
 OPT be the value of the optimum solution to **Set Cover**.

$$Z_{IP}^* = OPT.$$

△ We cannot solve this integer program efficiently (*why?*).

LP relaxation: a bound for the value of the IP

LP relaxation for **Set Cover**:

$$\begin{array}{ll} \min_{\mathbf{x} \geq \mathbf{0}} & \sum_{j=1}^m w_j x_j \\ \text{subject to} & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{array}$$

LP relaxation: a bound for the value of the IP

LP relaxation for **Set Cover**:

$$\begin{aligned} \min_{\mathbf{x} \geq \mathbf{0}} \quad & \sum_{j=1}^m w_j x_j \\ \text{subject to} \quad & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{aligned}$$

- ▶ Every feasible solution to the original IP is a feasible solution to the LP relaxation.
- ▶ The value of any feasible solution to the original IP is the same in the LP (the objectives are the same).
- ▶ Let Z_{LP}^* be the optimum value of the LP relaxation.

$$Z_{LP}^* \leq Z_{IP}^* = OPT$$

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Rounding the solution to the LP

LP relaxation for **Set Cover**:

$$\begin{aligned} \min_{\mathbf{x} \geq \mathbf{0}} \quad & \sum_{j=1}^n w_j x_j \\ \text{subject to} \quad & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{aligned}$$

- ▶ Let x^* be an optimal solution to the LP relaxation.
- ▶ Let $f_i = \#$ subsets S_j where element e_i appears.
- ▶ Let $f = \max_{1 \leq i \leq n} f_i$.
- ▶ Set

$$\hat{x}_j = \begin{cases} 1, & \text{if } x_j^* \geq 1/f \\ 0, & \text{if } x_j^* < 1/f \end{cases}$$

Rounding yields a feasible solution to the original IP

The collection of sets S_j with $\hat{x}_j = 1$ cover all the elements.

- ▶ Consider the optimal solution x^* for the LP relaxation.
- ▶ Fix any element e_i ; recall that e_i appears in f_i subsets.
- ▶ For simplicity, relabel these subsets as S_1, S_2, \dots, S_{f_i} . Then the optimal solution satisfies the constraint

$$x_1^* + x_2^* + \dots + x_{f_i}^* \geq 1$$

Let x_m^* be the maximum of $x_1^*, x_2^*, \dots, x_{f_i}^*$. Then

$$x_m^* \geq \frac{1}{f_i} \geq \frac{1}{f}$$

- ⇒ Our rounding procedure guarantees that, for every element e_i , at least one set S_j that *covers* e_i is chosen.

An f -approximation algorithm for Set Cover

How far is the solution obtained by the rounding procedure above from to the *optimal* solution to Set Cover?

- ▶ We do **not** know *OPT*!
- ▶ **But** we have a **bound** for it: the value Z_{LP}^* of the LP relaxation!

Recall that we set $\hat{x}_j = 1$ if and only if $x_j^* \geq 1/f$. Then

$$\begin{aligned}\sum_j w_j \hat{x}_j &\leq \sum_j w_j (f x_j^*) = f \sum_j w_j x_j^* \\ &= f \cdot Z_{LP}^* \leq f \cdot OPT\end{aligned}$$

Definition 8.

An α -approximation algorithm for an optimization problem is a polynomial-time algorithm that, for all instances of the problem, produces a solution whose value is within a factor of α of the value of the optimal solution.

Remark 1.

- ▶ α is the approximation ratio or approximation factor
- ▶ For *minimization* problems, $\alpha > 1$.
- ▶ For *maximization* problems, $\alpha < 1$.

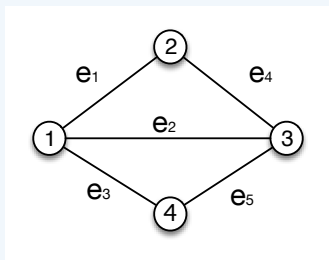
Example 1: the rounding procedure described on slide 53 gives an f -approximation algorithm for **Set Cover**:

- ▶ it can be completed in polynomial-time
- ▶ it always returns a solution whose value is at most f times the value of the optimal solution.

Remark: if an element appears in too many sets (e.g., $f = \Omega(n)$), this algorithm does not provide a good approximation guarantee.

Example 2: a 2-approximation algorithm for VC is a polynomial-time algorithm that always returns a solution whose value is at most twice the value of the optimal solution.

A 2-approximation algorithm for VC



- ▶ Let $E = \{e_1, \dots, e_m\}$ be the set of edges in the graph.
- ▶ Let S_j be the set of edges (ground elements) that are covered by vertex j .
- ▶ For every edge e_i , $f_i = 2$: e_i appears in exactly two subsets (*why?*).
- ▶ Hence $f = \max_{1 \leq i \leq m} f_i = 2$.