Algorithms for Data Science
CSOR W4246

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Outline

Review of last lecture

1. Data Compression
   - Symbol codes
   - Optimal lossless compression and prefix codes
   - Trees and prefix codes
2. Greedy algorithms
   - A greedy algorithm for optimal lossless compression using symbol codes: the Huffman algorithm

Today

Cache maintenance

- The offline problem
- An optimal algorithm for the offline problem: Farthest-in-Future (FF)
- Proof of optimality of FF
- The online problem

Today

1. The offline problem
2. An optimal algorithm for the offline problem: Farthest-into-Future (FF)
3. Proof of optimality of FF
4. Online problem

The setting

Input

- n, the number of pages in the main memory
- k, the size of the cache memory
- a sequence of m requests for memory pages $r_1, r_2, \ldots, r_m$

Example:

- $n = 3$
- $k = 2$
- $m = 7$
- sequence of requests: a, b, c, b, c, a, b

The setting (cont'd)

- To service a request, the corresponding page must be in the cache.

$\Rightarrow$ After the first k requests for distinct pages the cache is full.

- Cache miss: a request for a page that is not in the cache.
  - We must evict a page from the cache to bring in the requested page.

Assumption: a request is received and serviced within the same time step.

The problem

At each time step $1 \leq t \leq m$, we must decide which page (if any) to evict from the cache.

Definition 1 (Scheduling algorithm).

A schedule is a sequence of eviction decisions so that all m requests are serviced at time m. An algorithm that provides such a schedule is a scheduling algorithm.

Goal: find the schedule that minimizes the total number of cache misses.

Example

- # pages in main memory: $n = 3$
- cache size: $k = 2$
- sequence of $m = 7$ requests: a, b, c, b, c, a, b

time t:	1	2	3	4	5	6	7
requests:	a,	b,	c,	b,	c,	a,	b

Example

- # pages in main memory: $n = 3$
- cache size: $k = 2$
- sequence of $m = 7$ requests: a, b, c, b, c, a, b

time t:	1	2	3	4	5	6	7
requests:	a,	b,	c,	b,	c,	a,	b
eviction schedule S:	$-$,	$-$,	a,	$-$,	$-$,	c,	$-$
cache contents:	$\{a\}$	$\{a, b\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b, a\}$	$\{b, a\}$

- "$-$" stands for "no eviction"
- $S = \{-, -, a, -, -, c, -\}$ evicts a at time 3, c at time 6
- S incurs 2 cache misses *(can't do better here)*

Offline vs online problem

- Offline problem: the entire sequence of requests $\{r_1, r_2, \ldots, r_m\}$ is part of the input (known at time $t = 0$)
- Online problem (more natural): requests arrive one at a time; r_t must be serviced at time t, before future requests $r_{t+1}, \ldots, r_m$ are seen
- A scheduling algorithm for the online problem can only base its eviction decision at time t on
  1. the requests it has seen so far
  2. the eviction decisions it has made so far
- The optimal offline algorithm helps to understand the online problem *(coming up)*

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An optimal rule for the offline problem

Definition 2 (Farthest-into-Future).

FF: When the page requested at time i is not in the cache, evict from the cache the page that is needed the farthest into the future and bring in the requested page.

Notation: we will denote the schedule produced by this algorithm S_{FF}.

Why would S_{FF} be optimal?

Reduced schedules

Definition 3 (Reduced schedule).

A reduced schedule brings a page in the cache at time t only if

1. the page is requested at time t; *and*
2. the page is not already in the cache.

Remark 1.

1. *In a sense, a reduced schedule performs the least amount of work at every time step.*
2. *FF is a reduced schedule.*

There is an optimal *reduced* schedule

Fact 4.

We can transform a non-reduced schedule into a reduced one that is at least as good, that is, incurs at most the same number of evictions.

Remark 2.

- *The expensive memory operation is the eviction: even when no cache miss is incurred, we still count #evictions.*
- *In reduced schedules, #cache misses = # evictions.*
- *Given Fact 4, we can focus solely on reduced schedules.*

Proof of Fact 4

- Let S' be a schedule that is not reduced and solves an instance of cache maintenance.
- We will transform S' into a reduced schedule S
  - Time i, request $r_i \neq a$:
    - if S' evicts a page from the cache to bring in page a, not requested at time i
    - S *pretends* it brings in a but in fact does nothing
  - First time step $j > i$ such that $r_j = a$: S brings in a

  $\Rightarrow$ *charge* the cache miss of S' at time j to the eviction of S at the earlier time i
- Thus S performs at most as many evictions as S'.

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Optimality of FF

Claim 1.

Let S be a reduced schedule that makes the same eviction decisions as S_{FF} up to time $t = i$, that is, up to request i. Then there is a reduced schedule S' that

1. *makes the same eviction decisions as S_{FF} up to time $t = i + 1$, that is, up to request $i + 1$;*
2. *the total number of cache misses it incurs is no more than that incurred by S.*

Proposition 1.

The schedule S_{FF} provided by the Farthest-into-Future algorithm is optimal.

Proof of Proposition 1: case $i = 0$

Notation

- $cm(S) =$ total #cache misses of schedule S
- S^* is an optimal reduced schedule
- *Schedule* S_1 *follows schedule* S_2 *up to request* i if S_1 makes the same eviction decisions as S_2 up to the i-th request

$i = 0$: trivially, S^* follows S_{FF} up to request $i = 0$. By Claim 2, we can construct a reduced schedule S_1 such that

1. S_1 follows S_{FF} up to request $i = 1$
2. $cm(S_1) \leq cm(S^*)$.

Proof of Proposition 1: case $i > 0$

- $i = 1$: now S_1 is a reduced schedule that follows S_{FF} up to request $i = 1$. By Claim 2, we can construct a reduced schedule S_2 such that
  1. S_2 follows S_{FF} up to request $i = 2$
  2. $cm(S_2) \leq cm(S_1)$.
- $i = 2$: now S_2 is a reduced schedule that follows S_{FF} up to request $i = 2$. By Claim 2, we can construct a reduced schedule S_3 such that
  1. S_3 follows S_{FF} up to request $i = 3$
  2. $cm(S_3) \leq cm(S_2)$.

Proof of Proposition 1: $S_m = S_{FF}$

- Applying the claim for every $3 \leq i \leq m-1$, we obtain the reduced schedule S_m that
  1. follows S_{FF} up to time m
  2. $cm(S_m) \leq cm(S_{m-1})$.

  Tracing back all the inequalities, we obtain $cm(S_m) \leq cm(S^*)$.

- Finally, since S_m follows S_{FF} up to time m, $S_{FF} = S_m$.

  Hence $cm(S_{FF}) = cm(S_m) \leq cm(S^*)$.

  Thus S_{FF} is optimal.

Optimality of FF

Claim 2.

Let S be a reduced schedule that makes the same eviction decisions as S_{FF} up to time $t = i$, that is, up to request i. Then there is a reduced schedule S' that

1. *makes the same eviction decisions as S_{FF} up to time $t = i + 1$, that is, up to request $i + 1$;*
2. *incurs no more total cache misses than S does.*

Proposition 2.

The schedule S_{FF} provided by the Farthest-into-Future algorithm is optimal.

Proof of Claim 2

Notation:

- $cm(S) =$ total #cache misses of schedule S
- $C_i(S) =$ contents of the cache of schedule S at time i

Since S and S_{FF} have made the same scheduling decisions up to time i, at the end of time step i:

- the contents of their caches are identical, hence

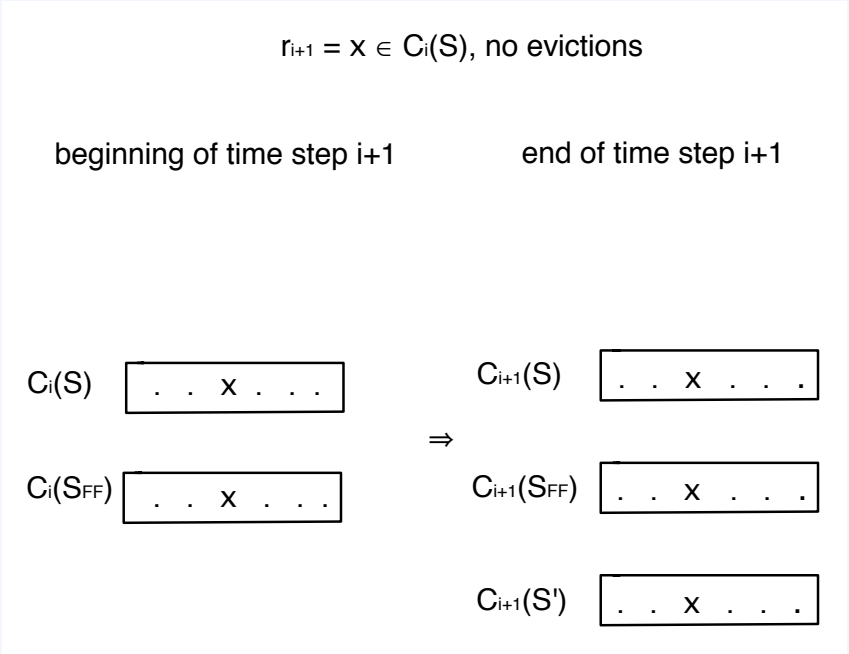
$$C_i(S) = C_i(S_{FF})$$

- so far, S has the same number of cache misses as S_{FF}

Suppose that at time $i+1$, page p is requested, hence $r_{i+1} = p$.

Proof of Claim 2, case 1: $r_{i+1} = x \in C_i(S)$

1. If $x \in C_i(S)$
   - $x \in C_i(S_{FF})$ since $C_i(S) = C_i(S_{FF})$
   - no cache miss for either schedule



   - Set $S' = S$; then
     1. S' follows S_{FF} up to time $i+1$ (S does!)
     2. $cm(S') \leq cm(S)$.

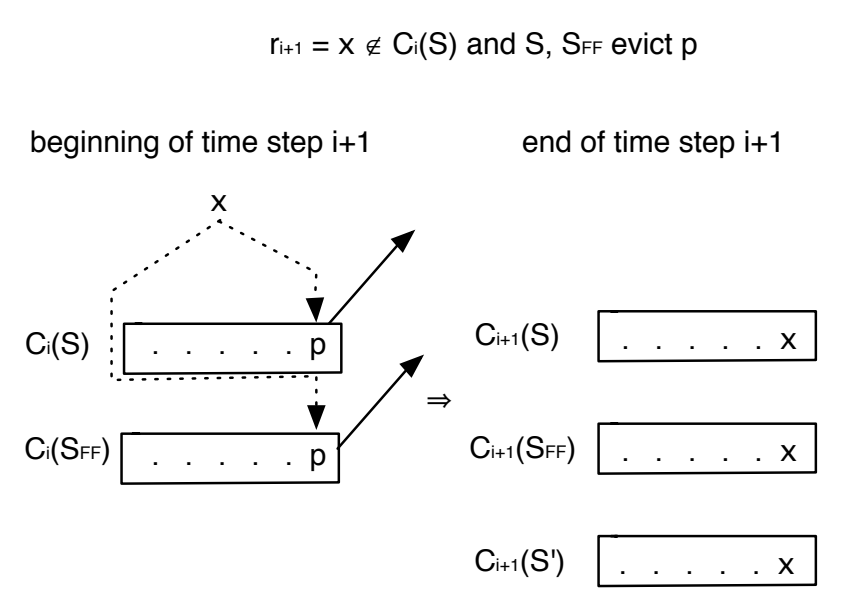
Proof of Claim 2, case 2: $r_{i+1} = x \notin C_i(S)$

2. If $x \notin C_i(S)$

   - x also not in $C_i(S_{FF})$ since $C_i(S) = C_i(S_{FF})$
   - both schedules must bring x in, hence incur a cache miss

   2.1: If S and S_{FF} both evict the same page p, set $S' = S$

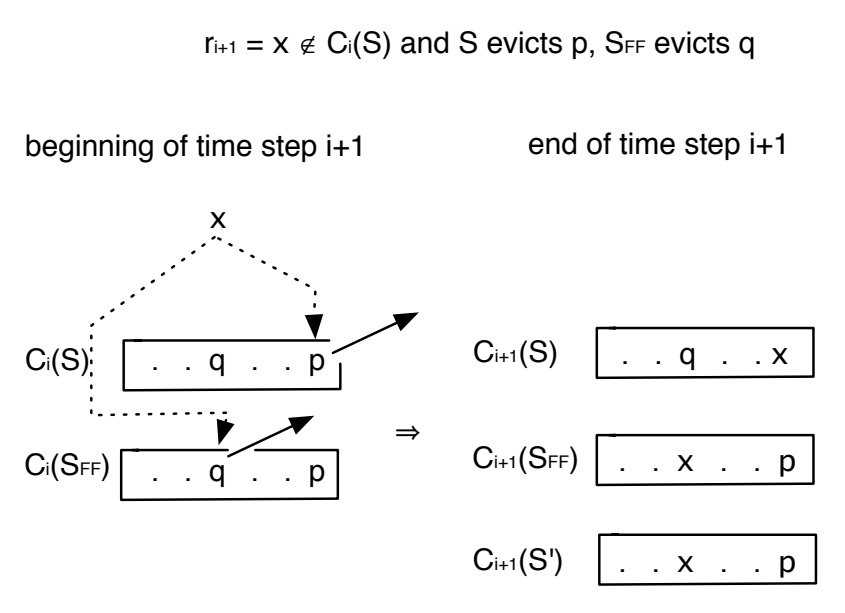
   1. S' follows S_{FF} up to time $i+1$ (since S does)
   2. $cm(S') \leq cm(S)$.



Proof of Claim 2, case 2.2: $r_{i+1} = x \notin C_i(S)$

2.2: If S evicts p but S_{FF} evicts q:

- By construction of S_{FF}, p must be requested later in the future than q.
- At the end of time step $i+1$, the cache contents for the two schedules will differ in exactly one item.



Proof of Claim 2, case 2.2: S evicts p, S_{FF} evicts q

At the end of time step $i+1$

- the cache of S contains q
- the cache of S_{FF} contains p
- the remaining $k-1$ items in both caches are the same
- thus

$$C_{i+1}(S_{FF}) = C_{i+1}(S) - \{q\} + \{p\}.$$

- Since we want S' to agree with S_{FF} up to time $i+1$, S' evicts σ from its cache as well. Hence

$$C_{i+1}(S') = C_{i+1}(S_{FF}) = C_{i+1}(S) - \{q\} + \{p\}.$$

Roadmap for case 2.2: S evicts p, S_{FF} evicts q

- At the end of time step $i+1$
  - the cache contents of S, S' differ in exactly one item
  - #cache misses of S = #cache misses of S'
- Want to ensure that S' will not incur more misses than S for $i+1 < t \leq m$.
- **Idea:** set $S' = S$ *as soon as* the cache contents of S, S' are the same again.

$\Rightarrow$ **Goal:** make $C_t(S')$ equal $C_t(S)$ for the earliest $t > i+1$ possible, while not incurring unnecessary misses.

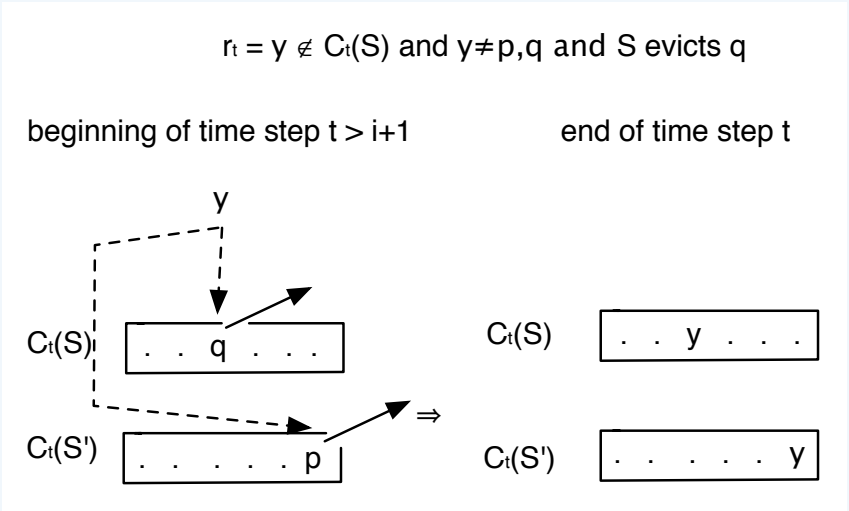
- Once $C_t(S') = C_t(S)$, set $S' = S$; if S' has not incurred more misses than S between steps $i+2$ and t, then $cm(S') \leq cm(S)$.

Case 2.2.1: $r_t = x \notin \{p, q\}$, $x \notin C_t(S)$, S evicts q

For all $t > i + 1$, S' follows S until one of the following happens for the first time:

2.2.1: $r_t = y \notin \{p, q\}$, **and** $y \notin C_t(S)$, **and** S evicts q.

Since $C_t(S)$ and $C_t(S')$ only differ in p, q, then $y \notin C_t(S')$. Set S' to evict p and bring in y. Then $C_t(S') = C_t(S)$!

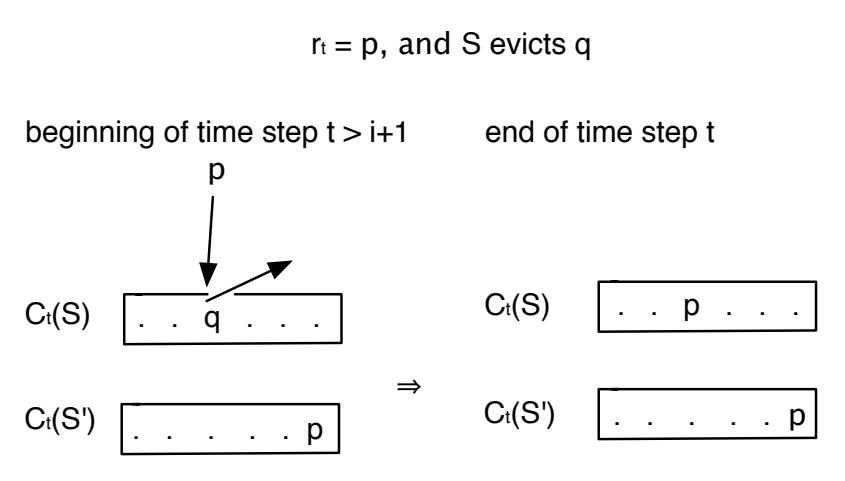


Set $S' = S$ henceforth: S' follows S_{FF} up to time $i + 1$ and $cm(S') \leq cm(S)$.

Case 2.2.2.1: $r_t = p$, S evicts q

2.2.2: $r_t = p$

2.2.2.1: If S evicts q, $C_t(S) = C_t(S')$!

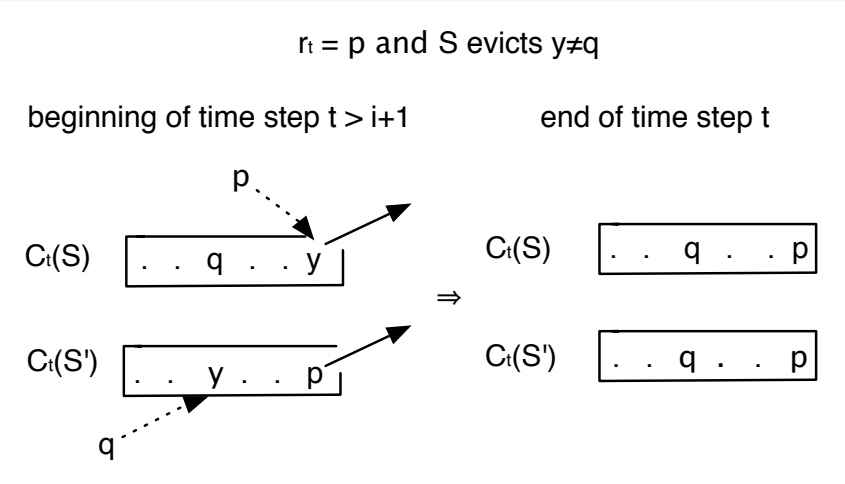


Set $S' = S$ henceforth: S' follows S_{FF} up to time $i + 1$ and $cm(S') < cm(S)$.

Case 2.2.2.2: $r_t = p$, S evicts $y \neq q$

2.2.2: $r_t = p$

2.2.2.2: If S evicts $y \neq q$ from its cache, then S' evicts y as well and brings in q. Then $C_t(S') = C_t(S)$.



Set $S' = S$ henceforth: S' follows S_{FF} up to time $i+1$ and $cm(S') < cm(S)$.

2.2.2.2: resolving the final issue (we're not done yet!)

- S' is no longer reduced: σ was brought in when there was no request for σ at time t (recall that $r_t = q$).
- Fortunately, we can use Fact 1 to transform S' into a reduced schedule $\overline{S}$ that
  - incurs at most the same number of evictions as S'
  - still follows S_{FF} up to time $i+1$: all the real evictions of the reduced S' will happen after time $i+1$!
- Hence we return $\overline{S}$ as the schedule that satisfies Claim 2.

2.2.3: $r_t = p$

Can't happen! S_{FF} evicted q and not p, hence

- p appears farther in the future than q
- one of cases i., ii. will happen first

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The online problem

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- Online problem (more natural): requests arrive one at a time; r_t must be serviced at time t, before $r_{t+1}, \ldots, r_m$ are seen
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- The optimal offline algorithm helps to understand the online problem.

The Least Recently Used principle

- The Least Recently Used (LRU) principle: evict the page that was requested the longest ago
- **Intuition**: a running program will generally keep accessing the things it's just been accessing (locality of reference)
- Essentially Farthest-into-Future (FF) reversed in time.
- LRU behaves well on average inputs.
- However an adversary can devise a specific sequence of online requests that will cause LRU to perform very badly compared to the optimal offline algorithm (*how?*).

Worst-case input to LRU

Example

- #pages in main memory: $n = 3$
- size of the cache: $k = 2$
- sequence of online requests

$$\underbrace{a, b, c}_{1}, \underbrace{a, b, c}_{2}, \ldots, \underbrace{a, b, c}_{k}$$

$\Rightarrow$ LRU: every request starting at time $t = 3$ is a miss, hence $7 = 3k - 2$ misses

$\Rightarrow$ FF: only $4 = (3k - 1)/2$ misses

Average-case performance of LRU

- Experimentally the best scheduling algorithms for the online problem are variants of LRU

- Competitive ratio: the worst-case ratio between the performance of the online algorithm over the performance of the optimal offline algorithm.