Analysis of Algorithms, I
CSOR W4231.002

Eleni Drinea
Computer Science Department

Columbia University

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Outline

Today

1. Review of last lecture
2. Integer Programming
3. Minimum-weight Set Cover
   - An integer programming formulation of `Set Cover`
   - The linear program relaxation
4. An approximation algorithm for `Set Cover`
   - Rounding the LP solution
   - An f-approximation algorithm for `Set Cover`

LPs in matrix-vector notation

We may rewrite any LP as follows (*think about it!*).

1. It is either a maximization or a minimization
2. All constraints are inequalities in the same direction
3. All variables are non-negative

This results in an LP of the following form

$$\begin{aligned} \max_{\mathbf{x} \geq \mathbf{0}} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \end{aligned}$$

The dual in matrix-vector notation

Then the dual is given as follows:

$$\begin{aligned} \min_{\mathbf{y} \geq \mathbf{0}} & \quad \mathbf{b}^T \mathbf{y} \\ \text{subject to} & \quad A^T \mathbf{y} \geq \mathbf{c} \end{aligned}$$

By construction, we know that any feasible solution to the dual is an upper bound for the primal (weak duality). Hence

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$$

What if the primal is unbounded?
What if the dual is unbounded?

Interpreting the dual LP (case study: max flow)

$$\max_{f_{ij} \geq 0} \sum_{j:(s,j)\in E} f_{sj}$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in E} f_{ij} - \sum_{j:(j,i)\in E} f_{ji} = \begin{cases} \sum_{j:(s,j)\in E} f_{sj}, & \text{if } i = s \\ -\sum_{j:(s,j)\in E} f_{sj}, & \text{if } i = t \\ 0, & \text{otherwise} \end{cases} \qquad (i \in V)$$

$$\text{and} \quad f_{ij} \leq c_{ij}, \qquad \text{for all } (i,j) \in E$$

- We want to maximize the flow out of source s.
- The entire flow must get routed to sink t.
- At intermediate nodes we must have flow conservation.

Max flow Dual LP

$$\begin{aligned} \min_{q \geq 0, p} \quad & \sum c_{ij} q_{ij} \\ \text{subject to} \quad & p_j - p_i \leq q_{ij} \quad ((i,j) \in E) \\ & p_t - p_s = 1 \end{aligned}$$

This is a minimum cut problem. Why?

Max flow Dual LP

$$
\begin{aligned}
\min_{q \geq 0, p} \quad & \sum c_{ij} q_{ij} \\
\text{subject to} \quad & p_j - p_i \leq q_{ij} \quad ((i,j) \in E) \\
& p_t - p_s = 1
\end{aligned}
$$

This is a minimum cut problem. Why?

At an optimal solution, nodes for which $p_i = 0$ are in S, and nodes for which $p_i = 1$ are in T, and (S,T) defines an s-t cut. We have

$$
q_{ij} = \begin{cases} 0 & \text{if nodes } i, j \text{ are in the same set} \\ 1 & \text{otherwise} \end{cases}
$$

so the objective value is the capacity of the (S,T) cut.

Max flow Dual LP

$$\min_{q \geq 0, p} \quad \sum c_{ij} q_{ij}$$

$$\text{subject to} \quad p_j - p_i \leq q_{ij} \quad ((i,j) \in E)$$

$$p_t - p_s = 1$$

This is a minimum cut problem. Why?

Strong duality

$$\textit{maximum flow} = \textit{minimum cut}$$

Today

Integer Programming

`Integer programming (IP(D))`: Given a system of linear inequalities in n variables and m constraints with integer coefficients and a integer target value k, does it have an integer solution of value k?

- Applications: production planning, scheduling trains, etc.

Example:

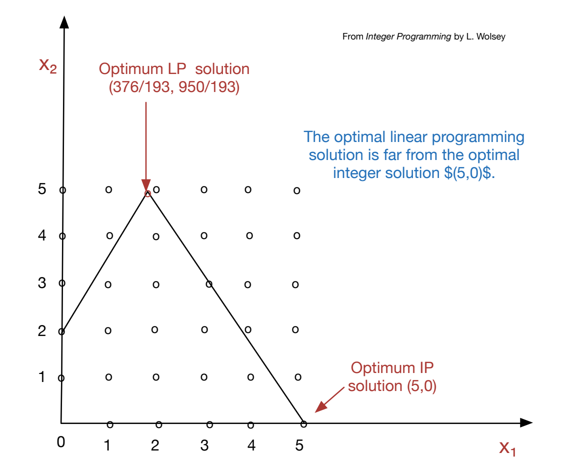
$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbf{Z}^n \end{aligned}$$

Here A is an $m \times n$ matrix, $\mathbf{b} \in \mathbf{R}^m$, $\mathbf{c} \in \mathbf{R}^n$, $\mathbf{x}$ is an integer vector with n components.

What does the set of feasible solutions look like?

Rounding the LP is often insufficient

$$\begin{aligned}
\max_{x_1 \geq 0, x_2 \geq 0} \quad & 1.00x_1 + 0.64x_2 \\
\text{subject to} \quad & 50x_1 + 31x_2 \leq 250 \\
& 3x_1 - 2x_2 \geq -4 \\
& x_1, x_2 \text{ integer}
\end{aligned}$$



Is `IP(D)` *hard?*

- `IP(D)` is in $\mathcal{NP}$.
- We can quickly solve LPs with several thousands of variables and constraints but there exist integer programs with 10 variables and 10 constraints that are very hard to solve.

Is `IP(D)` *hard?*

- `IP(D)` is in $\mathcal{NP}$.
- We can quickly solve LPs with several thousands of variables and constraints but there exist integer programs with 10 variables and 10 constraints that are very hard to solve.

- This is not too surprising: integer programs restricted to solutions $\mathbf{x} \in \{0,1\}^n$ model **yes**/**no** decisions, which are generally hard.

- To formalize this intuition, we will reduce an $\mathcal{NP}$-complete problem to `IP(D)`.

Integer Programs for `Vertex Cover` and `IS`

First we formulate integer programs for two $\mathcal{NP}$-hard problems.

IP for `Independent Set`:

$$\begin{aligned} \text{max} \quad & \sum_{i=0}^{n} x_i \\ \text{subject to} \quad & x_i + x_j \leq 1, \quad \text{for every } (i,j) \in E \\ & x_i \in \{0,1\}, \quad \text{for every } i \in V \end{aligned}$$

IP for `Vertex Cover`:

$$\begin{aligned} \text{min} \quad & \sum_{i=0}^{n} x_i \\ \text{subject to} \quad & x_i + x_j \geq 1, \quad \text{for every } (i,j) \in E \\ & x_i \in \{0,1\}, \quad \text{for every } i \in V \end{aligned}$$

IP(D) is $\mathcal{NP}$-complete

Claim 1.

VC(D) $\leq_P$ IP(D)

Proof.

Reduction from arbitrary instance $(G = (V, E), k)$ of VC(D) to the following integer program with target value k:

$$\sum_{i=1}^{n} x_i \leq k$$

$$\begin{aligned} \text{subject to} \quad & x_i + x_j \geq 1, \quad && \text{for every } (i, j) \in E \\ & x_i \in \{0, 1\}, \quad && \text{for every } i \in V \end{aligned}$$

Equivalence of the instances is straightforward. □

Similar problems with very different complexities *(new)*

$\mathcal{NP}$	$\mathcal{P}$
max cut	min cut
longest path	shortest path
3D matching	matching
Hamiltonian cycle	Euler cycle
3-colorability	2-colorability
3-SAT	2-SAT
LCS of n sequences	LCS of 2 sequences
integer programming	linear programming

The theory of integer and linear programming and duality can guide the design of approximation algorithms for hard problems.

Today

`Minimum-weight Set Cover`

Input

- a set $E = \{e_1, e_2, \ldots, e_n\}$ of n elements
- a collection of subsets of these elements $S_1, S_2, \ldots, S_m$, where each $S_j \subseteq E$
- a non-negative weight w_j for every subset S_j

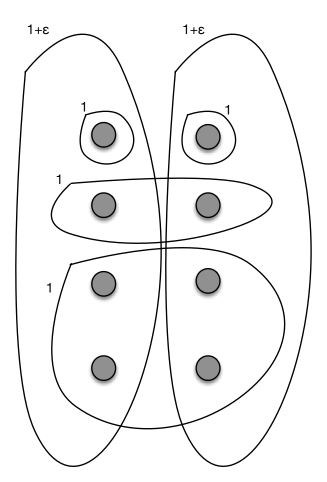
Output

A minimum-weight collection of subsets that cover all of E.

In symbols: find an $I \subseteq \{1, \ldots, m\}$ such that $\cup_{i \in I} S_i = E$ and $\sum_{i \in I} w_i$ is minimum.

(`Unweighted Set Cover`: $w_j = 1$ for all j)

Example instance of `Set Cover`



$n = 8$ ground elements, $m = 6$ subsets with weights
$w_1 = w_2 = w_3 = w_4 = 1$, $w_5 = w_6 = 1 + \epsilon$.

Motivation: detect computer viruses

Motivation: detect features of viruses that do not occur in typical applications

- **Ground elements:** computer viruses ($n \approx 150$)
- **Sets:** labelled by some three-byte sequence occurring in these viruses but not occurring in typical computer applications ($m \approx 21000$); each set consisted of all the viruses that contained the three-byte sequence
- **Objective**: output a small number of such sequences (much smaller than 150) that *cover* all known viruses

Reduction via generalization

Claim 2.

`Set-Cover(D)` *is* $\mathcal{NP}$*-complete.*

Proof.

Reduction from `VC(D)`. Input instance: $(G = (V, E), k)$.

- Set $E = \{e_1, \ldots, e_m\}$ to be the set of ground elements we want to *cover*.
- For every vertex j, set S_j to be the set of edges (ground elements) that are incident to –hence *covered* by– vertex j.
- Set $w_j = 1$ for all $1 \leq j \leq n$.

Equivalence of instances: input graph has a vertex cover of size k if and only if E has a set cover of weight k. $\square$

Designing the integer program for `Set Cover`

Variables: we introduce one variable per set S_j; intuitively,

$$x_j = \begin{cases} 1, & \text{if } S_j \text{ is included in the solution} \\ 0, & \text{otherwise} \end{cases}$$

Constraints: ensure that every element is *covered*:

for every element e_i, at least one of the sets S_j
containing e_i appears in the final solution

Objective function: minimize the sum of the weights of the sets included in the solution

An integer programming formulation of `Set Cover`

Integer program for `Set Cover`:

$$
\begin{aligned}
\min \quad & \sum_{i=0}^{n} w_j x_j \\
\text{subject to} \quad & \sum_{j : e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \\
& x_j \in \{0, 1\}, \qquad \text{for every } 1 \leq j \leq m
\end{aligned}
$$

An integer programming formulation of `Set Cover`

Integer program for `Set Cover`:

$$\begin{aligned} \min \quad & \sum_{i=0}^{n} w_j x_j \\ \text{subject to} \quad & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \\ & x_j \in \{0, 1\}, \quad \text{for every } 1 \leq j \leq m \end{aligned}$$

Let Z^*_{IP} be the optimum value of this integer program; OPT be the value of the optimum solution to `Set Cover`.

$$Z^*_{IP} = OPT.$$

△ We cannot solve this integer program efficiently *(why?)*.

LP relaxation: a bound for the value of the IP

LP relaxation for `Set Cover`:

$$\begin{aligned} \min_{\mathbf{x}\geq \mathbf{0}} \quad & \sum_{i=0}^{n} w_j x_j \\ \text{subject to} \quad & \sum_{j:e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{aligned}$$

LP relaxation: a bound for the value of the IP

LP relaxation for `Set Cover`:

$$\begin{aligned} \min_{\mathbf{x}\geq\mathbf{0}} \quad & \sum_{i=0}^{n} w_j x_j \\ \text{subject to} \quad & \sum_{j:e_i\in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{aligned}$$

- Every feasible solution to the original IP is a feasible solution to the LP relaxation.
- The value of any feasible solution to the original IP is the same in the LP (the objectives are the same).
- Let Z^*_{LP} be the optimum value of the LP relaxation.

$$Z^*_{LP} \leq Z^*_{IP} = OPT$$

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4 An approximation algorithm for `Set Cover`
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Rounding the solution to the LP

LP relaxation for `Set Cover`:

$$\begin{aligned} \min_{\mathbf{x}\geq\mathbf{0}} \quad & \sum_{i=0}^{n} w_j x_j \\ \text{subject to} \quad & \sum_{j:e_i\in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{aligned}$$

- Let x^* be an optimal solution to the LP relaxation.
- Let $f_i = \#$ subsets S_j where element e_i appears.
- Let $f = \max_{1\leq i\leq n} f_i$.
- Set

$$\hat{x}_j = \begin{cases} 1, & \text{if } x_j^* \geq 1/f \\ 0, & \text{if } x_j^* < 1/f \end{cases}$$

Rounding yields a feasible solution to the original IP

The collection of sets S_j with $\hat{x}_j = 1$ cover all the elements.

- Consider the optimal solution x^* for the LP relaxation.
- Fix any element e_i; recall that e_i appears in f_i subsets.
- For simplicity, relabel these subsets as $S_1, S_2, \ldots, S_{f_i}$. Then the optimal solution satisfies the constraint

$$x_1^* + x_2^* + \ldots + x_{f_i}^* \geq 1$$

Let x_m^* be the maximum of $x_1^*, x_2^*, \ldots, x_{f_i}^*$. Then

$$x_m^* \geq \frac{1}{f_i} \geq \frac{1}{f}$$

$\Rightarrow$ Our rounding procedure guarantees that, for every element e_i, at least one set S_j that *covers* e_i is chosen.

An f-approximation algorithm for `Set Cover`

How far is the solution obtained by the rounding procedure above from to the optimal solution to `Set Cover`*?*

- We do **not** know OPT!
- **But** we have a bound for it: the value Z^*_{LP} of the LP relaxation!

Recall that we set $\hat{x}_j = 1$ if and only if $x^*_j \geq 1/f$. Then

$$\begin{aligned} \sum_j w_j \hat{x}_j &\leq \sum_j w_j (f x^*_j) = f \sum_j w_j x^*_j \\ &= f \cdot Z^*_{LP} \leq f \cdot OPT \end{aligned}$$

Approximation algorithms

Definition 1.

An α-approximation algorithm for an optimization problem is a polynomial-time algorithm that, for all instances of the problem, produces a solution whose value is within a factor of α of the value of the optimal solution.

Remark 1.

- α *is the* approximation ratio *or* approximation factor
- *For minimization problems,* $\alpha > 1$.
- *For maximization problems,* $\alpha < 1$.

Examples

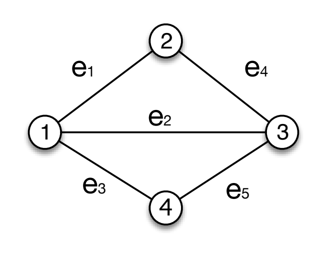
Example 1: the rounding procedure described on slide 30 gives an f-approximation algorithm for `Set Cover`:

- it can be completed in polynomial-time
- it always returns a solution whose value is at most f times the value of the optimal solution.

Remark: if an element appears in too many sets (e.g., $f = \Omega(n)$), this algorithm does not provide a good approximation guarantee.

Example 2: a 2-approximation algorithm for `VC` is a polynomial-time algorithm that always returns a solution whose value is at most twice the value of the optimal solution.

A 2-approximation algorithm for VC



- Let $E = \{e_1, \ldots, e_m\}$ be the set of edges in the graph.
- Let S_j be the set of edges (ground elements) that are covered by vertex j.
- For every edge e_i, $f_i = 2$: e_i appears in exactly two subsets *(why?)*.
- Hence $f = \max_{1 \leq i \leq m} f_i = 2$.