

Analysis of Algorithms, I

CSOR W4231

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Shortest paths in weighted graphs (Bellman-Ford, Floyd-Warshall)

- 1 Shortest paths in graphs with non-negative edge weights (Dijkstra's algorithm)
 - Implementations
 - Graphs with **negative** edge weights: why Dijkstra fails
- 2 Single-source shortest paths (negative edges): Bellman-Ford
 - A DP solution
 - An alternative formulation of Bellman-Ford
- 3 All-pairs shortest paths (negative edges): Floyd-Warshall

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Graphs with non-negative weights

Input

- ▶ a weighted, directed graph $G = (V, E, w)$; function $w : E \rightarrow \mathbb{R}^+$ assigns non-negative real-valued weights to edges;
- ▶ a source (**origin**) vertex $s \in V$.

Output: for every vertex $v \in V$

1. the length of a shortest s - v path;
2. a shortest s - v path.

Dijkstra's algorithm (Input: $G = (V, E, w), s \in V$)

Output: arrays $dist, prev$ with n entries such that

1. $dist(v)$ stores the length of the shortest $s-v$ path
2. $prev(v)$ stores the node before v in the shortest $s-v$ path

At all times, maintain a set S of nodes for which the distance from s has been determined.

- ▶ Initially, $dist(s) = 0, S = \{s\}$.
- ▶ Each time, add to S the node $v \in V - S$ that
 1. has an edge from some node in S ;
 2. minimizes the following quantity among all nodes $v \in V - S$

$$d(v) = \min_{u \in S: (u,v) \in E} \{dist(u) + w(u,v)\}$$

- ▶ Set $prev(v) = u$.

Implementation

Dijkstra-v1($G = (V, E, w), s \in V$)

Initialize(G, s)

$S = \{s\}$

while $S \neq V$ **do**

 Select a node $v \in V - S$ with at least one edge from S so that

$$d(v) = \min_{u \in S, (u,v) \in E} \{dist[u] + w(u, v)\}$$

$S = S \cup \{v\}$

$dist[v] = d(v)$

$prev[v] = u$

end while

Initialize(G, s)

for $v \in V$ **do**

$dist[v] = \infty$

$prev[v] = NIL$

end for

$dist[s] = 0$

Improved implementation (I)

Idea: Keep a **conservative overestimate** of the true length of the shortest s - v path in $dist[v]$ as follows: when u is added to S , **update** $dist[v]$ for all v with $(u, v) \in E$.

Dijkstra-v2($G = (V, E, w), s \in V$)

Initialize(G, s)

$S = \emptyset$

while $S \neq V$ **do**

 Pick u so that $dist[u]$ is minimum among all nodes in $V - S$

$S = S \cup \{u\}$

for $(u, v) \in E$ **do**

 Update(u, v)

end for

end while

Update(u, v)

if $dist[v] > dist[u] + w(u, v)$ **then**

$dist[v] = dist[u] + w(u, v)$

$prev[v] = u$

end if

Improved implementation (II): binary min-heap

Idea: Use a **priority queue implemented as a binary min-heap**: store vertex u with key $dist[u]$. Required operations: **Insert**, **ExtractMin**; **DecreaseKey** for **Update**; each takes $O(\log n)$ time.

Dijkstra-v3($G = (V, E, w), s \in V$)

Initialize(G, s)

$Q = \{V; dist\}$

$S = \emptyset$

while $Q \neq \emptyset$ **do**

$u = \text{ExtractMin}(Q)$

$S = S \cup \{u\}$

for $(u, v) \in E$ **do**

 Update(u, v)

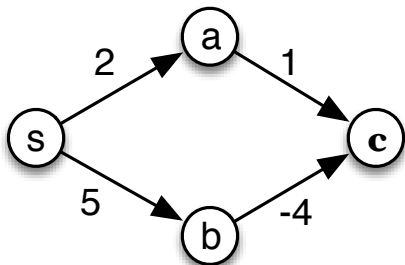
end for

end while

Running time: $O(n \log n + m \log n) = O(m \log n)$

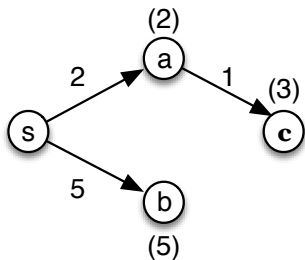
When is Dijkstra-v3() better than Dijkstra-v2()?

Example graph with negative edge weights

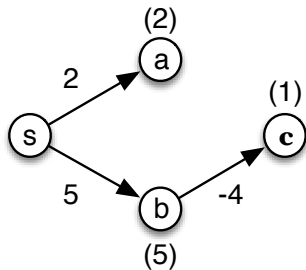


Dijkstra's output and correct output for example graph

Dijkstra's output



Correct shortest paths

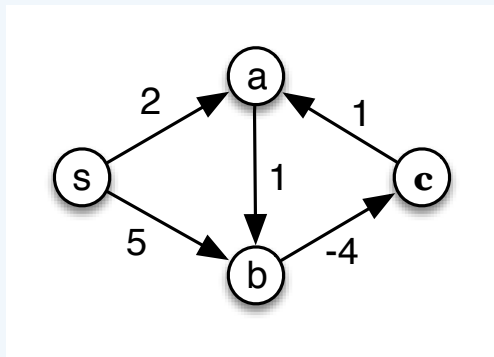


Dijkstra's algorithm will first include *a* to *S* and then *c*, thus missing the shorter path from *s* to *b* to *c*.

Negative edge weights: why Dijkstra's algorithm fails

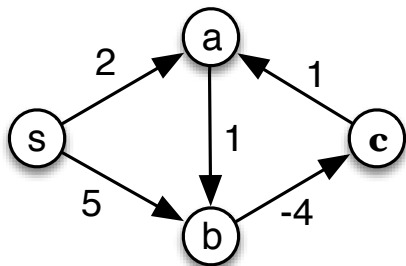
- ▶ Intuitively, a path may start on long edges but then compensate along the way with short edges.
- ▶ Formally, in the proof of correctness of the algorithm, the last statement about P does not hold anymore: even if the length of path P_v is smaller than the length of the subpath $s \rightarrow x \rightarrow y$, negative edges on the subpath $y \rightarrow v$ may now result in P being *shorter* than P_v .

Bigger problems in graphs with negative edges?



$dist(a) = ?$

Bigger problems in graphs with negative edges?



1. $dist(v)$ goes to $-\infty$ for every v on the cycle (a, b, c, a)
 2. **no** solution to shortest paths when negative cycles
- \Rightarrow need to **detect** negative cycles

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Single-source shortest paths, negative edge weights

Input: weighted directed graph $G = (V, E, w)$ with $w : E \rightarrow \mathbb{R}$;
a source (**origin**) vertex $s \in V$.

Output:

1. If G has a negative cycle reachable from s , answer
“**negative cycle in G** ”.
2. Else, compute for every $v \in V$
 - 2.1 the length of a shortest s - v path;
 - 2.2 a shortest s - v path.

Properties of shortest paths

Suppose the problem **has** a solution for an input graph.

- ▶ *Can there be negative cycles in the graph?*
- ▶ *Can there be positive cycles in the graph?*
- ▶ *Can the shortest paths contain positive cycles?*
- ▶ *Consider a shortest s - t path; are its subpaths shortest? In other words, does the problem exhibit optimal substructure?*

Towards a DP solution

Key observation: if there are no negative cycles, a path cannot become shorter by traversing a cycle.

Fact 1.

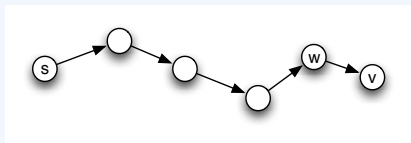
If G has no negative cycles, then there is a shortest s - v path that is simple, thus has at most $n - 1$ edges.

Fact 2.

The shortest paths problem exhibits optimal substructure.

Facts 1 and 2 suggest a DP solution.

Subproblems



Let

$OPT(i, v) =$ cost of a shortest s - v path with *at most* i edges

Consider a shortest s - v path using at most i edges.

- ▶ If the path uses at most $i - 1$ edges, then

$$OPT(i, v) = OPT(i - 1, v).$$

- ▶ If the path uses i edges, then

$$OPT(i, v) = \min_{x:(x,v) \in E} \{OPT(i - 1, x) + w(x, v)\}.$$

Recurrence

Let

$OPT(i, v)$ = cost of a shortest s - v path using *at most* i edges

Then

$$OPT(i, v) = \begin{cases} 0 & , \text{ if } i = 0, v = s \\ \infty & , \text{ if } i = 0, v \neq s \\ \min \left\{ \begin{array}{l} OPT(i-1, v) \\ \min_{x:(x,v) \in E} \{OPT(i-1, x) + w(x, v)\} \end{array} \right\} & , \text{ if } i > 0 \end{cases}$$

Pseudocode

$n \times n$ dynamic programming table M such that
 $M[i, v] = OPT(i, v)$.

Bellman-Ford($G = (V, E, w), s \in V$)

for $v \in V$ **do**

$M[0, v] = \infty$

end for

$M[0, s] = 0$

for $i = 1, \dots, n - 1$ **do**

for $v \in V$ (*in any order*) **do**

$$M[i, v] = \min \left\{ \begin{array}{l} M[i - 1, v] \\ \min_{x:(x,v) \in E} \left\{ M[i - 1, x] + w(x, v) \right\} \end{array} \right\}$$

end for

end for

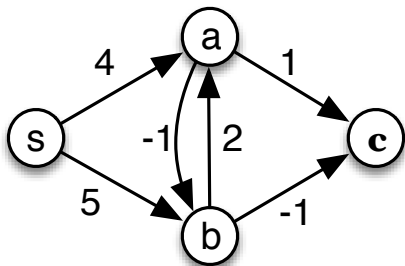
Running time & Space

- ▶ **Running time:** $O(nm)$
- ▶ **Space:** $\Theta(n^2)$ —can be improved (*coming up*)
- ▶ To reconstruct actual shortest paths, also keep array *prev* of size n such that

$prev[v]$ = predecessor of v in current shortest s - v path.

Example of Bellman-Ford

Compute shortest s - v paths in the graph below, for all $v \in V$.



Improving space requirements

Only need two rows of M at all times.

△ Actually, only need one (see Remark 1)! Thus drop the index i from $M[i, v]$ and only use it as a counter for #repetitions.

$$M[v] = \min \left\{ M[v], \min_{x:(x,v) \in E} \{ M[x] + w(x, v) \} \right\}$$

Remark 1.

Throughout the algorithm, $M[v]$ is the length of some s - v path. After i repetitions, $M[v]$ is no larger than the length of the current shortest s - v path with at most i edges.

Early termination condition: if at some iteration i **no** value in M changed, then stop (*why?*)

An alternative way to view Bellman-Ford



- ▶ Let $P = (s = v_0, v_1, v_2, \dots, v_k = v)$ be a shortest s - v path.
- ▶ Then P can contain at most $n - 1$ edges.
- ▶ *How can we correctly compute $\text{dist}(v)$ on this path?*

Key observations about subroutine `Update(u, v)`

Recall subroutine `Update` from Dijkstra's algorithm:

$$\text{Update}(u, v) : \text{dist}(v) = \min\{\text{dist}(v), \text{dist}(u) + w(u, v)\}$$

Fact 3.

Suppose u is the last node before v on the shortest s - v path, and suppose $\text{dist}(u)$ has been correctly set. The call `Update(u, v)` returns the correct value for $\text{dist}(v)$.

Fact 4.

*No matter how many times `Update(u, v)` is performed, it will never make $\text{dist}(v)$ too small. That is, `Update` is a **safe** operation: performing few extra updates can't hurt.*

Performing the correct sequence of updates

Suppose we update the edges on the shortest path P **in the order they appear on the path** (though not necessarily consecutively). Hence we update

$$(s, v_1), (v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v).$$

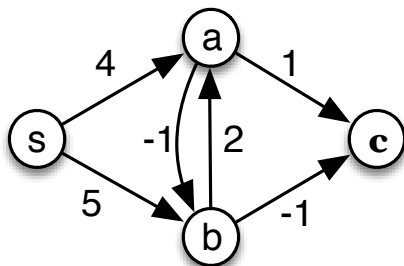
This sequence of updates correctly computes $dist(v_1)$, $dist(v_2)$, \dots , $dist(v)$ (by induction and Fact 3).

How can we guarantee that this specific sequence of updates occurs?

A concrete example

Consider the shortest s - b path, which uses edges (s, a) , (a, b) .

*How can we guarantee that our algorithm will update these two edges **in this order**? (More updates in between are allowed.)*



Bellman-Ford algorithm

Update all m edges in the graph, $n - 1$ times in a row!

- ▶ By Fact 4, it is ok to update an edge several times in between.
- ▶ All we need is to update the edges on the path in this **particular order**. This is guaranteed if we update all edges $n - 1$ times in a row.

Pseudocode

We will use `Initialize` and `Update` from Dijkstra's algorithm.

`Initialize`(G, s)

for $v \in V$ **do**

$dist[v] = \infty$

$prev[v] = NIL$

end for

$dist[s] = 0$

`Update`(u, v)

if $dist[v] > dist[u] + w(u, v)$ **then**

$dist[v] = dist[u] + w(u, v)$

$prev[v] = u$

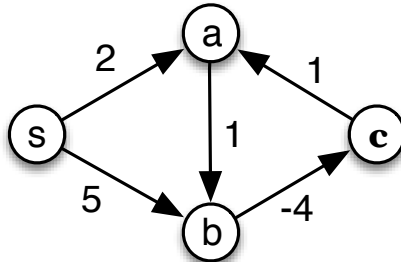
end if

Bellman-Ford

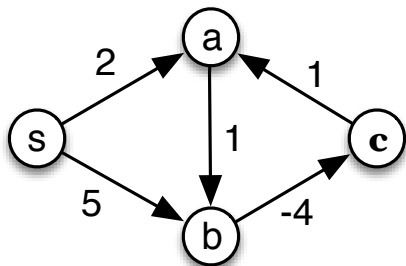
```
Bellman-Ford( $G = (V, E, w), s$ )  
  Initialize( $G, s$ )  
  for  $i = 1, \dots, n - 1$  do  
    for  $(u, v) \in E$  do  
      Update( $u, v$ )  
    end for  
  end for
```

Running time? Space?

Detecting negative cycles



Detecting negative cycles



1. $dist(v)$ goes to $-\infty$ for every v on the cycle.
2. Any shortest s - v path can have at most $n - 1$ edges.
3. Update all edges n times (instead of $n - 1$): if $dist(v)$ changes **for any** $v \in V$, then there is a negative cycle.

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All pairs shortest-paths

- ▶ **Input:** a directed, weighted graph $G = (V, E, w)$ with real edge weights
- ▶ **Output:** an $n \times n$ matrix D such that

$$D[i, j] = \text{length of shortest path from } i \text{ to } j$$

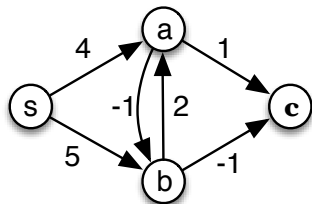
Solving all pairs shortest-paths

1. Straightforward solution: run Bellman–Ford once for every vertex ($O(n^2m)$ time).
2. Improved solution: Floyd-Warshall's dynamic programming algorithm ($O(n^3)$ time).

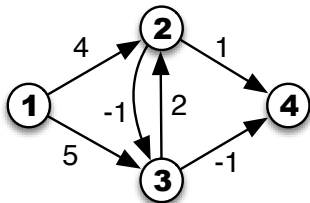
Towards a DP formulation

- ▶ Consider a shortest s - t path P .
- ▶ This path uses some **intermediate** vertices: that is, if $P = (s, v_1, v_2, \dots, v_k, t)$, then v_1, \dots, v_k are intermediate vertices.
- ▶ For simplicity, relabel the vertices in V as $\{1, 2, 3, \dots, n\}$ and consider a shortest i - j path where **intermediate vertices may only be from $\{1, 2, \dots, k\}$** .
- ▶ **Goal:** compute the length of a shortest i - j path for every pair of vertices (i, j) , using $\{1, 2, \dots, n\}$ as intermediate vertices.

Example



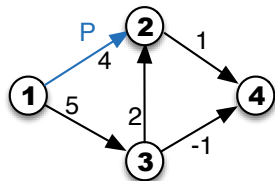
Rename {s, a, b, c} as **{1, 2, 3, 4}**



Examples of shortest paths

Shortest **(1, 2)**-path using $\{\}$ or $\{1\}$ is P .

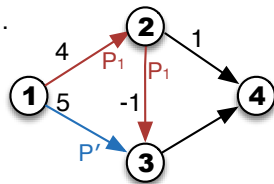
Shortest **(1, 2)**-path using $\{1,2,3,4\}$ is P .



Shortest **(1, 3)**-path using $\{\}$ or $\{1\}$ is P' .

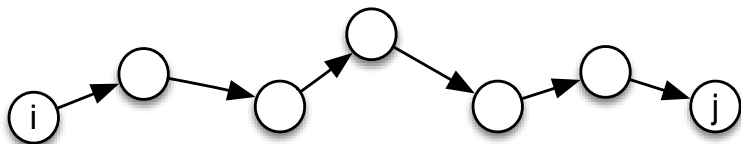
Shortest **(1, 3)**-path using $\{1,2\}$ or $\{1,2,3\}$ is P_1 .

Shortest **(1, 3)**-path using $\{1,2,3,4\}$ is P_1 .



A shortest i - j path using nodes from $\{1, \dots, k\}$

Consider a shortest i - j path P where intermediate nodes may only be from the set of nodes $\{1, 2, \dots, k\}$.

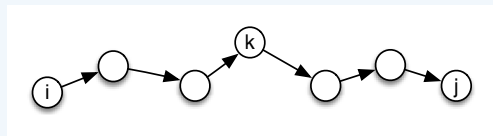


Fact: any subpath of P must be shortest itself.

A useful observation

Focus on the last node k from the set $\{1, 2, \dots, k\}$. Either

1. P completely avoids k : then a shortest i - j path with intermediate nodes from $\{1, \dots, k\}$ is the same as a shortest i - j path with intermediate nodes from $\{1, \dots, k - 1\}$.
2. Or, k is an intermediate node of P .



Decompose P into an i - k subpath P_1 and a k - j subpath P_2 .

- i. P_1, P_2 are shortest subpaths themselves.
- ii. All intermediate nodes of P_1, P_2 are from $\{1, \dots, k - 1\}$.

Subproblems

Let

$OPT_k(i, j)$ = cost of shortest $i - j$ path P using $\{1, \dots, k\}$ as intermediate vertices

1. Either k does not appear in P , hence

$$OPT_k(i, j) = OPT_{k-1}(i, j)$$

2. Or, k appears in P , hence

$$OPT_k(i, j) = OPT_{k-1}(i, k) + OPT_{k-1}(k, j)$$

Hence

$$OPT_k(i, j) = \begin{cases} w(i, j) & , \text{ if } k = 0 \\ \min \left\{ \begin{array}{l} OPT_{k-1}(i, j) \\ OPT_{k-1}(i, k) + OPT_{k-1}(k, j) \end{array} \right. & , \text{ if } k \geq 1 \end{cases}$$

We want $OPT_n(i, j)$.

Time/space requirements?

Floyd-Warshall on example graph

Let $D_k[i, j] = OPT_k(i, j)$.

$$D_0 =$$

0	4	5	∞
∞	0	-1	1
∞	2	0	-1
∞	∞	∞	0

$$D_1 =$$

0	4	5	∞
∞	0	-1	1
∞	2	0	-1
∞	∞	∞	0

$$D_2 =$$

0	4	3	5
∞	0	-1	1
∞	2	0	-1
∞	∞	∞	0

$$D_3 =$$

0	4	3	2
∞	0	-1	-2
∞	2	0	-1
∞	∞	∞	0

Space requirements

- ▶ A single $n \times n$ dynamic programming table D , initialized to $w(i, j)$ (the adjacency matrix of G).
- ▶ Let $\{1, \dots, k\}$ be the set of intermediate nodes that may be used for the shortest i - j path.
- ▶ After the k -th iteration, $D[i, j]$ contains the length of some i - j path that is no larger than the length of the shortest i - j path using $\{1, \dots, k\}$ as intermediate nodes.

The Floyd-Warshall algorithm

```
Floyd-Warshall( $G = (V, E, w)$ )  
  for  $k = 1$  to  $n$  do  
    for  $i = 1$  to  $n$  do  
      for  $j = 1$  to  $n$  do  
         $D[i, j] = \min(D[i, j], D[i, k] + D[k, j])$   
      end for  
    end for  
  end for
```

- ▶ Running time: $O(n^3)$
- ▶ Space: $\Theta(n^2)$