Analysis of Algorithms, I

CSOR W4231

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Linear programming: duality

Outline

Today

Why linear programming?

1. Vast range of applications
   - Resource allocation
   - Production planning
   - Military strategy forming
   - Graph theoretic problems
   - Error correction
   - ...
2. Establish the existence of polynomial-time (efficient) algorithms
3. Guide the design of approximation algorithms for computationally hard problems *(coming up in the next two weeks)*
4. Duality provides a unifying view of seemingly unrelated results and is useful in algorithm design

An introductory example: profit maximization

A boutique chocolatier has two products:

- an assortment of chocolates
- an assortment of truffles

Their profit is

1. $1 per box of chocolates
2. $6 per box of truffles

They can produce a total of at most 400 boxes per day.
The daily demand for these products is limited

1. at most 200 boxes of chocolates per day
2. at most 300 boxes of truffles per day

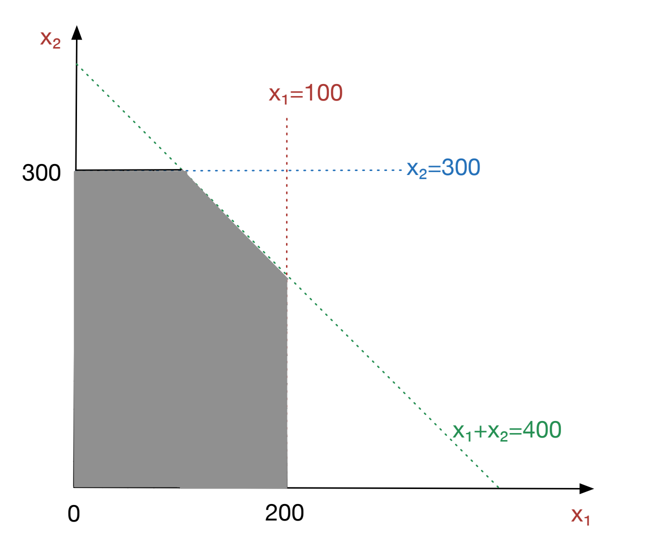
What are the optimal levels of production?

The LP for profit maximization

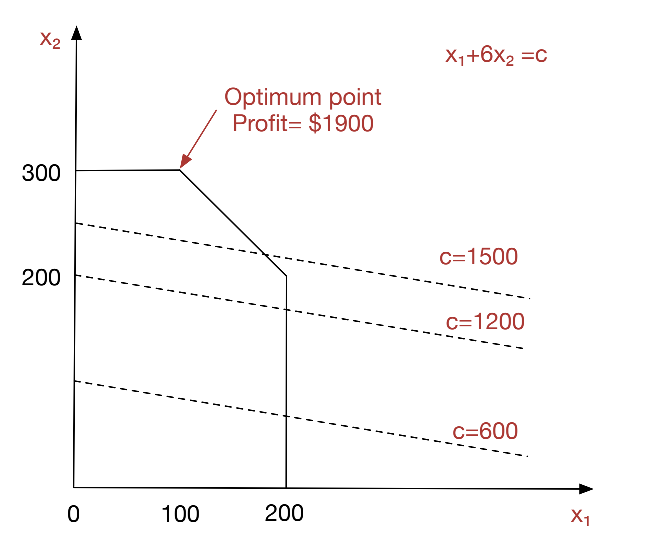
Thus we have the following LP for the chocolatier's profit

$$
\begin{aligned}
\max_{x_1 \geq 0, x_2 \geq 0} \quad & x_1 + 6x_2 \\
\text{subject to} \quad & x_1 \leq 200 \\
& x_2 \leq 300 \\
& x_1 + x_2 \leq 400
\end{aligned}
$$

The geometry of the solution: feasible region



The geometry of the solution: objective function



Feasible region and optimal solution

Fact 1.

The optimum is achieved at a vertex of the feasible region.

Exceptions

1. The linear program is infeasible
   - e.g., $x \leq 1$, $x \geq 2$
2. The optimum value is unbounded

$$\max_{x_1 \geq 0, x_2 \geq 0} \quad x_1 + x_2$$

Can the feasible region be unbounded, yet the objective function have a unique optimum value?

An alternative proof that $1900 is optimal

- Multiply the first inequality by 0
- Multiply the second inequality by 5
- Multiply the third inequality by 1
- Add the new inequalities; then

$$x_1 + 6x_2 \leq 1900$$

$\Rightarrow$ the objective function cannot exceed 1900!

$\Rightarrow$ thus we indeed found the optimal solution

Where did we get the multipliers $0, 5$ *and* 1*?*

Upper bounding the objective function

The constraints themselves can help us derive an upper bound.

Multiplier	Inequality		
y_1	x_1	$\leq$	200
y_2	x_2	$\leq$	300
y_3	$x_1 + x_2$	$\leq$	400

- Multipliers y_i must be non-negative (*why?*)

Add the multiplied inequalities together:

$$y_1 x_1 + y_2 x_2 + y_3(x_1 + x_2) \leq 200 y_1 + 300 y_2 + 400 y_3$$

An upper bound for the objective

We want to upper bound the original objective

$$1x_1 + 6x_2$$

using the linear combination

$$\begin{aligned} & y_1x_1 + y_2x_2 + y_3(x_1 + x_2) \leq 200y_1 + 300y_2 + 400y_3 \\ \Rightarrow \quad & (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3 \qquad (1) \end{aligned}$$

Since $x_1, x_2 \geq 0$, if we constrain $y_1 + y_3 \geq 1$ and $y_2 + y_3 \geq 6$, then the right-hand side in (1) is an upper bound for our objective.

The dual LP

- *What is the best possible upper bound for our objective?* Minimize equation (1) subject to constraints on y_1, y_2, y_3.
- This is yet another LP!

$$\begin{aligned} \min_{y_1 \geq 0, y_2 \geq 0, y_3 \geq 0} \quad & 200y_1 + 300y_2 + 400y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \end{aligned}$$

This new LP is called the dual of the original, which is called the primal.

Weak duality

- By construction, any feasible solution for the dual LP is an upper bound on the original primal LP.
- Let V_P be the optimal objective value for the primal (a maximization)
- Let V_D be the optimal objective value for the dual (a minimization)

Theorem 2 (Weak Duality).

$$V_P \leq V_D$$

Strong duality and consequences

For LPs with bounded optima

Theorem 3 (Strong Duality).

$$V_P = V_D$$

- We can alternatively solve the dual to find the optimal objective value.
- An optimal dual solution can be used to derive an optimal primal solution (complementary slackness).
- The dual may have structure making it easier to solve at scale (e.g., via parallel optimization).

Today

1. Recap
2. Taking the dual of an LP
3. Examples of formulating LPs
4. Interpreting the dual LP

LPs in matrix-vector notation

We may rewrite any LP as follows (*think about it!*).

1. It is either a maximization or a minimization
2. All constraints are inequalities in the same direction
3. All variables are non-negative

This results in an LP of the following form

$$\begin{aligned} \max_{\mathbf{x} \geq \mathbf{0}} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \end{aligned}$$

The dual in matrix-vector notation

Then the dual is given as follows:

$$\begin{aligned} \min_{\mathbf{y} \geq \mathbf{0}} \quad & \mathbf{b}^T \mathbf{y} \\ \text{subject to} \quad & A^T \mathbf{y} \geq \mathbf{c} \end{aligned}$$

By construction, we know that any feasible solution to the dual is an upper bound for the primal (weak duality). Hence

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$$

What if the primal is unbounded?
What if the dual is unbounded?

Feasibility vs Optimality

Finding a feasible solution of a linear program is generally computationally as difficult as finding an optimal solution.

For example, any feasible solution (restricted to $\mathbf{x}$) to the following LP is an optimal solution to the primal on slide 17 (the objective here is immaterial).

$$\begin{aligned} \max_{\mathbf{x}\geq\mathbf{0},\mathbf{y}\geq\mathbf{0}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y} \end{aligned}$$

Textbook dualization recipe

Note that $\mathbf{b} \in \mathcal{R}^m$, $\mathbf{c} \in \mathcal{R}^n$, $A \in \mathcal{R}^{m \times n}$

	Primal LP	Dual LP
Variables	$x_1, \ldots, x_n$	$y_1, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective	$\max \mathbf{c}^T \mathbf{x}$	$\min \mathbf{b}^T \mathbf{y}$
Constraints	$x_i \geq 0$	i-th constraint has $\geq$
	$x_i \leq 0$	$\leq$
	$x_i \in \mathcal{R}$	$=$
	j-th constraint has $\leq$	$y_j \geq 0$
	$\geq$	$y_j \leq 0$
	$=$	$y_j \in \mathcal{R}$

7-step dualization you can remember!

Example LP

$$\max_{x_1 \geq 0, x_2 \leq 0, x_3} \quad c_1x_1 + c_2x_2 + c_3x_3 \tag{2}$$

$$\text{subject to} \quad a_1x_1 + x_2 + x_3 \leq b_1 \tag{3}$$

$$x_1 + a_2x_2 = b_2 \tag{4}$$

$$a_3x_3 \geq b_3 \tag{5}$$

Step 1: work with a minimization problem

If necessary, rewrite the objective as a **minimization**.

$$\min_{x_1 \geq 0, x_2 \leq 0, x_3} -c_1 x_1 - c_2 x_2 - c_3 x_3 \tag{6}$$

Step 2: rewrite the LP in a convenient form

Rewrite each inequality constraint (except for the special constraints under the min) as a "$\leq$", and rearrange each constraint so that the right-hand side is 0.

$$
\begin{aligned}
\min_{x_1 \geq 0, x_2 \leq 0, x_3} \quad & -c_1 x_1 - c_2 x_2 - c_3 x_3 \\
\text{subject to} \quad & a_1 x_1 + x_2 + x_3 - b_1 \leq 0 && (7) \\
& x_1 + a_2 x_2 - b_2 = 0 && (8) \\
& -a_3 x_3 + b_3 \leq 0 && (9)
\end{aligned}
$$

Step 3: introducing the dual variables

Define

- a **non-negative** dual variable (*multiplier*) for each inequality constraint (except for those under the min)
- an **unrestricted** dual variable for each equality constraint.

Intuitively, this ensures that the direction of the inequality does not change by multiplying it with the dual variable (the sign of the multiplier does not matter for an equality).

We introduce $y_1 \geq 0, y_2, y_3 \geq 0$ for constraints (7), (8) and (9) respectively.

Step 4: maximizing the sum of everything

For each constraint, eliminate it and add the term

$$\text{(dual variable)} \cdot \text{(left-hand side of the constraint)}$$

to the objective. Maximize the result over the dual variables.

$$\begin{aligned} \max_{y_1 \geq 0, y_2, y_3 \geq 0} \quad \min_{x_1 \geq 0, x_2 \leq 0, x_3} \quad & -c_1 x_1 - c_2 x_2 - c_3 x_3 \\ + \quad & y_1(a_1 x_1 + x_2 + x_3 - b_1) && (10) \\ + \quad & y_2(x_1 + a_2 x_2 - b_2) && (11) \\ + \quad & y_3(-a_3 x_3 + b_3) && (12) \end{aligned}$$

Step 5: grouping terms by primal variables

Rewrite the objective so that it consists of

1. terms involving **only** dual variables
2. terms of the form

   (primal variable) · (expression with dual variables)

$$
\begin{aligned}
\max_{y_1 \geq 0, y_2, y_3 \geq 0} \quad \min_{x_1 \geq 0, x_2 \leq 0, x_3} \quad & -b_1 y_1 - b_2 y_2 + b_3 y_3 \\
+ \quad & x_1(a_1 y_1 + y_2 - c_1) \qquad (13) \\
+ \quad & x_2(y_1 + a_2 y_2 - c_2) \qquad (14) \\
+ \quad & x_3(y_1 - a_3 y_3 - c_3) \qquad (15)
\end{aligned}
$$

Step 6: eliminating primal variables to get the dual LP

Remove each term of the form

$$\text{(primal variable)} \cdot \text{(expression with dual variables)}$$

from the objective and add a constraint of the form

- expression ≥ 0, if the primal variable is non-negative.
- expression $= 0$, if the primal variable is unconstrained.
- expression ≤ 0, if the primal variable is non-positive.

$$\begin{aligned}
\max_{y_1 \geq 0, y_2, y_3 \geq 0} \quad & -b_1 y_1 - b_2 y_2 + b_3 y_3 \\
\text{subject to} \quad & a_1 y_1 + y_2 - c_1 \geq 0 && (16) \\
& y_1 + a_2 y_2 - c_2 \leq 0 && (17) \\
& y_1 - a_3 y_3 - c_3 = 0 && (18)
\end{aligned}$$

Step 7

If the original LP was a maximization rewritten as a minimization in Step 1, rewrite the result of the previous step as a minimization.

$$\min_{y_1 \geq 0, y_2, y_3 \geq 0} \quad b_1 y_1 + b_2 y_2 - b_3 y_3$$

$$\text{subject to} \quad a_1 y_1 + y_2 - c_1 \geq 0 \tag{19}$$

$$y_1 + a_2 y_2 - c_2 \leq 0 \tag{20}$$

$$y_1 - a_3 y_3 - c_3 = 0 \tag{21}$$

Exercise

Use the 7-step procedure for dualization described in slides 22 to 28 to find the dual of the following LP.

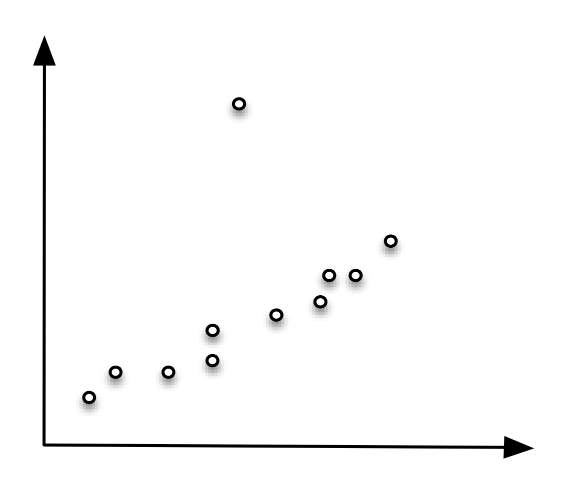
$$\begin{aligned} \max_{\mathbf{x} \geq \mathbf{0}} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & C\mathbf{x} \leq \mathbf{d} \end{aligned}$$

Then take the dual of this LP to confirm that it indeed gives the primal LP.

Today

Fitting a line

Given a data set of n points (x_i, y_i) on the plane, find a line of *best* fit.



Minimizing least squares errors

1. Least squares: find a line $y = ax + b$ that minimizes

$$\sum_{i=1}^{n} (ax_i + b - y_i)^2.$$

Solution:

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}$$

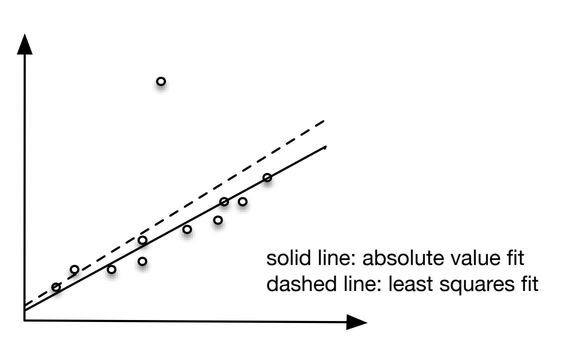
$$b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

△ *Outliers can affect the resulting line significantly.*

Minimizing the absolute values of all errors

2. Another method to find a line of best fit that is less sensitive to few outliers is to find the line $y = ax + b$ that minimizes the absolute values of all errors:

$$\sum_{i=1}^{n} |ax_i + b - y_i|$$



An LP for minimizing absolute values of all errors

$$
\begin{aligned}
\min_{\mathbf{e} \geq \mathbf{0}} \quad & \sum_{i=1}^{n} e_i \\
\text{subject to} \quad & e_i \geq a x_i + b - y_i, && \text{for } i = 1, 2, \ldots, n \\
& e_i \geq -(a x_i + b - y_i) && \text{for } i = 1, 2, \ldots, n
\end{aligned}
$$

Remark 1.

Absolute values can often be handled by introducing extra variables or extra constraints.

Today

Max flow LP

$$\max_{f_{ij} \geq 0} \quad \sum_{j:(s,j)\in E} f_{sj}$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in E} f_{ij} - \sum_{j:(j,i)\in E} f_{ji} = \begin{cases} \sum_{j:(s,j)\in E} f_{sj}, & \text{if } i = s \\ -\sum_{j:(s,j)\in E} f_{sj}, & \text{if } i = t \\ 0, & \text{otherwise} \end{cases} \qquad (i \in V)$$

$$\text{and} \quad f_{ij} \leq c_{ij}, \qquad \text{for all } (i,j) \in E$$

- We want to maximize the flow out of source s.
- The entire flow must get routed to sink t.
- At intermediate nodes we must have flow conservation.

Max flow Dual LP

$$\min_{q \geq 0, p} \quad \sum c_{ij} q_{ij}$$

$$\text{subject to} \quad p_j - p_i \leq q_{ij} \quad ((i,j) \in E)$$

$$p_t - p_s = 1$$

This is a minimum cut problem. Why?

Max flow Dual LP

$$\begin{aligned} \min_{q \geq 0, p} \quad & \sum c_{ij} q_{ij} \\ \text{subject to} \quad & p_j - p_i \leq q_{ij} \quad ((i,j) \in E) \\ & p_t - p_s = 1 \end{aligned}$$

This is a minimum cut problem. Why?

At an optimal solution, nodes for which $p_i = 0$ are in S, and nodes for which $p_i = 1$ are in T, and (S, T) defines an s-t cut. We have

$$q_{ij} = \begin{cases} 0 & \text{if nodes } i, j \text{ are in the same set} \\ 1 & \text{otherwise} \end{cases}$$

so the objective value is the capacity of the (S, T) cut.

Max flow Dual LP

$$\min_{q \geq 0, p} \quad \sum c_{ij} q_{ij}$$

$$\text{subject to} \quad p_j - p_i \leq q_{ij} \quad ((i,j) \in E)$$

$$p_t - p_s = 1$$

This is a minimum cut problem. Why?

Strong duality

$$\textit{maximum flow} = \textit{minimum cut}$$

Shortest s-t path LP

The single-source single-destination shortest-paths problem, henceforth referred to as s-t shortest-path problem, can be formulated as an LP.

$$\min_{f \geq 0} \quad \sum_{f \geq 0} w_{ij} f_{ij}$$

$$\text{subject to} \quad \sum_{j:(i,j)\in E} f_{ij} - \sum_{j:(j,i)\in E} f_{ji} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases} \qquad (i \in V)$$

Shortest s-t path LP

The single-source single-destination shortest-paths problem, henceforth referred to as s-t shortest-path problem, can be formulated as an LP.

$$\min_{f \geq 0} \quad \sum_{f \geq 0} w_{ij} f_{ij}$$

$$\text{subject to} \quad \sum_{j:(i,j) \in E} f_{ij} - \sum_{j:(j,i) \in E} f_{ji} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases} \qquad (i \in V)$$

- Constraints specify flow out of each node.
- Flow starts at source s, must end at sink t.
- Flow minimizes total weight (i.e., finds shortest path).

Shortest s-t path LP

The single-source single-destination shortest-paths problem, henceforth referred to as s-t shortest-path problem, can be formulated as an LP.

$$\min_{f \geq 0} \quad \sum_{f \geq 0} w_{ij} f_{ij}$$

$$\text{subject to} \quad \sum_{j:(i,j)\in E} f_{ij} - \sum_{j:(j,i)\in E} f_{ji} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases} \qquad (i \in V)$$

Fact

With flow constraints, there is an integer optimal solution f^* to the LP where $f^*_{ij} \in \{0, 1\}$ for each edge $(i, j) \in E$.

Shortest s-t path dual LP

$$
\begin{aligned}
\max_{p} \quad & p_t - p_s \\
\text{subject to} \quad & p_j - p_i \leq w_{ij} \quad (i,j) \in E
\end{aligned}
$$

- Imagine nodes i and j are attached by a string of length w_{ij}.
- If we pull nodes s and t as far apart as possible, the strings that are taut are those that are part of the shortest path.

Shortest s-t path dual LP

$$\begin{aligned} \max_{p} \quad & p_t - p_s \\ \text{subject to} \quad & p_j - p_i \leq w_{ij} \quad (i,j) \in E \end{aligned}$$

Strong duality

$$\textit{minimum path length} = \textit{maximum tension}$$