Analysis of Algorithms, I
CSOR W4231.002

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Satisfiability problems: SAT, 3SAT, Circuit-SAT

Outline

1. Complexity classes
   - The class $\mathcal{NP}$
   - The class of $\mathcal{NP}$-complete problems
2. *Satisfiability:* a fundamental $\mathcal{NP}$-complete problem
3. The art of proving $\mathcal{NP}$-completeness
   - Circuit-SAT $\leq_P$ SAT
   - `3SAT` $\leq_P$ `IS(D)`

Today

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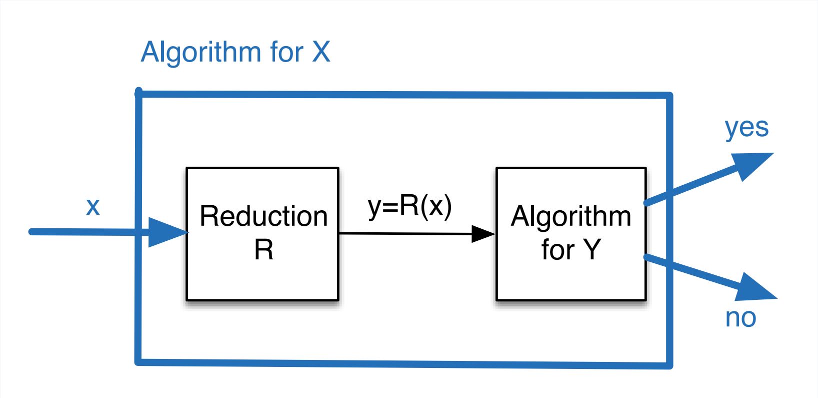
Diagram of a reduction $\mathtt{X} \leq_P Y$

$\mathtt{X}$, $\mathtt{Y}$ are computational problems; R is a polynomial time transformation from input x to y so that x, y are equivalent



We used reductions

- as a means to design efficient algorithms
- for arguing about the relative hardness of problems

Decision versions of optimization problems

An optimization problem `X` may be transformed into a roughly equivalent problem with a **yes/no** answer, called the decision version `X(D)` of the optimization problem, by

1. supplying a target value for the quantity to be optimized;
2. asking whether this value can be attained.

Examples:

- `IS(D)`: given a graph G and an integer k, does G have an independent set of size k?
- `VC(D)`: given a graph G and an integer k, does G have a vertex cover of size k?

The complexity class $\mathcal{P}$ and beyond

Definition 1.

$\mathcal{P}$ is the set of problems that can be **solved** by polynomial-time algorithms.

Beyond $\mathcal{P}$*?*

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Beyond $\mathcal{P}$?

Problems like `IS(D)` and `VC(D)`:

- No polynomial time algorithm has been found despite significant effort, so we don't believe they are in $\mathcal{P}$.
- *Is there anything positive we can say about such problems?*

The class $\mathcal{NP}$

Definition 2.

An efficient certifier (or *verification algorithm*) B for a problem `X(D)` is a polynomial-time algorithm that

1. takes **two** input arguments, the instance x and the *short* certificate t (both encoded as binary strings)
2. there is a polynomial $p(\cdot)$ so that for every string x, we have $x \in$ `X(D)` if and only if there is a string t such that $|t| \leq p(|x|)$ and $B(x, t) =$ **yes**.

Note that existence of the certifier B does not provide us with any efficient way to solve `X(D)`! (*why?*)

Definition 3.

We define $\mathcal{NP}$ to be the set of decision problems that have an efficient certifier.

$\mathcal{P}$ vs $\mathcal{NP}$

Fact 4.

$$\mathcal{P} \subseteq \mathcal{NP}$$

Proof.

Let $\texttt{X(D)}$ be a problem in $\mathcal{P}$.

- There is an efficient algorithm A that solves $\texttt{X(D)}$, that is, $A(x) =$ **yes** if and only if $x \in \texttt{X(D)}$.
- To show that $\texttt{X(D)} \in \mathcal{NP}$, we need exhibit an efficient certifier B that takes two inputs x and t and answers **yes** if and only if $x \in \texttt{X(D)}$.
- The algorithm B that on inputs x, t, simply discards t and simulates $A(x)$ is such an efficient certifier.

$\square$

$\mathcal{P}$ vs $\mathcal{NP}$

$$\mathcal{P} \quad = \quad \mathcal{NP} \quad ?$$

$\mathcal{P}$ vs $\mathcal{NP}$

$$\mathcal{P} \quad = \quad \mathcal{NP} \quad ?$$

- *Arguably the biggest question in theoretical CS*
- *We do not think so: finding a solution should be harder than checking one, especially for hard problems...*

Why would $\mathcal{NP}$ contain more problems than $\mathcal{P}$?

- Intuitively, the hardest problems in $\mathcal{NP}$ are the least likely to belong to $\mathcal{P}$.

- How do we identify the hardest problems?

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- How do we identify the hardest problems?

The notion of reduction is useful again.

Definition 5 ($\mathcal{NP}$-complete problems:).

A problem `X(D)` is $\mathcal{NP}$-complete if

1. `X(D)` $\in \mathcal{NP}$, and
2. for all `Y` $\in \mathcal{NP}$, `Y` $\leq_P$ `X`.

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1. $\texttt{X(D)} \in \mathcal{NP}$ and
2. for all $\texttt{Y} \in \mathcal{NP}$, $\texttt{Y} \leq_P \texttt{X}$.

Fact 6.

Suppose $\texttt{X}$ is $\mathcal{NP}$-complete. Then $\texttt{X}$ is solvable in polynomial time (i.e., $\texttt{X} \in \mathcal{P}$) if and only if $\mathcal{P} = \mathcal{NP}$.

Why we should care whether a problem is $\mathcal{NP}$-complete

- If a problem is $\mathcal{NP}$-complete it is among the least likely to be in $\mathcal{P}$: it is in $\mathcal{P}$ if and only if $\mathcal{P} = \mathcal{NP}$.

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- Therefore, from an algorithmic perspective, we need to stop looking for efficient algorithms for the problem.

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- Therefore, from an algorithmic perspective, we need to stop looking for efficient algorithms for the problem.

Instead we have a number of options

1. approximation algorithms, that is, algorithms that return a solution within a provable guarantee from the optimal
2. exponential algorithms practical for small instances
3. work on interesting special cases
4. study the average performance of the algorithm
5. examine heuristics (algorithms that work well in practice, yet provide no theoretical guarantees regarding how close the solution they find is to the optimal one)

How do we show that a problem is $\mathcal{NP}$-complete?

Suppose we had an $\mathcal{NP}$-complete problem X.

To show that another problem Y is $\mathcal{NP}$-complete, we only need show that

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2. $X \leq_P Y$

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Why?

Fact 7 (Transitivity of reductions).

If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.

We know that for all $A \in \mathcal{NP}$, $A \leq_P X$. By Fact 15, $A \leq_P Y$. Hence Y is $\mathcal{NP}$-complete.

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So, *if* we had a first $\mathcal{NP}$-complete problem X, discovering a new problem Y in this class would require an *easier* kind of reduction: just reduce X to Y (instead of reducing *every* problem in $\mathcal{NP}$ to Y!).

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The first $\mathcal{NP}$-complete problem

Theorem 7 (Cook-Levin).

Circuit SAT is $\mathcal{NP}$-complete.

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Boolean logic

Syntax of Boolean expressions

- Boolean variable x: a variable that takes values from $\{0, 1\}$ (equivalently, $\{F, T\}$, standing for **False**, **True**).
- Suppose you are given a set of n boolean variables $\{x_1, x_2, \ldots, x_n\}$.
- Boolean connectives: logical AND $\wedge$, logical OR $\vee$ and logical NOT $\neg$
- Boolean expression or Boolean formula: boolean variables connected by boolean connectives
- **Notational convention**: ϕ is a boolean formula

Boolean expressions

A boolean expression may be any of the following

1. A boolean variable, e.g., x_i.
2. The negation of a Boolean expression ϕ, denoted by $\neg\phi_1$ or $\overline{\phi_1}$.
3. The disjunction (logical OR) of two Boolean expressions in parentheses $(\phi_1 \vee \phi_2)$.
4. The conjunction (logical AND) of two Boolean expressions in parentheses $(\phi_1 \wedge \phi_2)$.

Properties of Boolean expressions

Basic properties of Boolean expressions (associativity, commutativity, distribution laws)

1. $\neg\neg\phi \equiv \phi$
2. $(\phi_1 \vee \phi_2) \equiv (\phi_2 \vee \phi_1)$
3. $(\phi_1 \wedge \phi_2) \equiv (\phi_2 \wedge \phi_1)$
4. $((\phi_1 \vee \phi_2) \vee \phi_3) \equiv (\phi_1 \vee (\phi_2 \vee \phi_3))$
5. $((\phi_1 \wedge \phi_2) \wedge \phi_3) \equiv (\phi_1 \wedge (\phi_2 \wedge \phi_3))$
6. $((\phi_1 \vee \phi_2) \wedge \phi_3) \equiv ((\phi_1 \wedge \phi_3) \vee (\phi_2 \wedge \phi_3))$
7. $((\phi_1 \wedge \phi_2) \vee \phi_3) \equiv ((\phi_1 \vee \phi_3) \wedge (\phi_2 \vee \phi_3))$
8. $\neg(\phi_1 \vee \phi_2) \equiv (\neg\phi_1 \wedge \neg\phi_2)$
9. $\neg(\phi_1 \wedge \phi_2) \equiv (\neg\phi_1 \vee \neg\phi_2)$
10. $\phi_1 \vee \phi_1 \equiv \phi_1$
11. $\phi_1 \wedge \phi_1 \equiv \phi_1$

Conjunctive Normal Form (CNF)

A literal ℓ_i is a variable or its negation.

Definition 8.

A Boolean formula ϕ is in CNF if it consists of conjunctions of clauses each of which is a disjunction of literals.

- In symbols, a formula ϕ with m clauses is in CNF if

$$\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$$

  and each clause C_i is the disjunction of a number of literals

$$\ell_1 \vee \ell_2 \vee \ldots \vee \ell_k$$

- **Example:** $n = 3$, $m = 2$, $\phi = (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_3)$

Remark: we will henceforth work with formulas in CNF.

Semantics of boolean formulas

- Let $X = \{x_1, \ldots, x_n\}$.
- A truth assignment for X is an assignment of truth values from $\{0, 1\}$ to each x_i.
  - So a truth assignment is a function $v : X \to \{0, 1\}$.
  - It is implied that $\overline{x_i}$ obtains value opposite from x_i.
  - **Example:** $X = \{x_1, x_2, x_3\}$
    - Truth assignment for X: $x_1 = 1, x_2 = x_3 = 0$
- A truth assignment causes a boolean formula to receive a value from $\{0, 1\}$.
  - **Example:** $\phi = (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_3)$
    - The above truth assignment causes ϕ to evaluate to 0.

Satisfying truth assignments

- A truth assignment satisfies a clause if it causes the clause to evaluate to 1.
  - **Example:** $\phi = (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_3)$
    Then $x_1 = x_2 = 1, x_3 = 0$ satisfies both clauses in ϕ.
- A truth assignment satisfies a formula in CNF if it satisfies **every** clause in the formula.
  - **Example:** $x_1 = x_2 = 1, x_3 = 0$ satisfies the above ϕ.
    But $x_1 = 1, x_2 = x_3 = 0$ does **not** satisfy ϕ.
- A formula ϕ is satisfiable if it has a satisfying truth assignment.
  - **Example:** the above ϕ is satisfiable; a *certificate* of its satisfiability is the truth assignment $x_1 = x_2 = 1, x_3 = 0$.

Satisfiability (SAT) and 3SAT

Definition 9 (SAT).

Given a formula ϕ in CNF with n variables and m clauses, is ϕ satisfiable?

Satisfiability ($\mathtt{SAT}$) and $\mathtt{3SAT}$

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Given a formula ϕ in CNF with n variables and m clauses, is ϕ satisfiable?

A convenient (and not easier) variant of $\mathtt{SAT}$ requires that every clause consists of exactly three literals.

Definition 10 ($\mathtt{3SAT}$).

Given a formula ϕ in CNF with n variables and m clauses such that each clause has exactly 3 literals, is ϕ satisfiable?

Are these problems hard?

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Theorem 11.

`SAT`, `3SAT` *are* $\mathcal{NP}$*-complete.*

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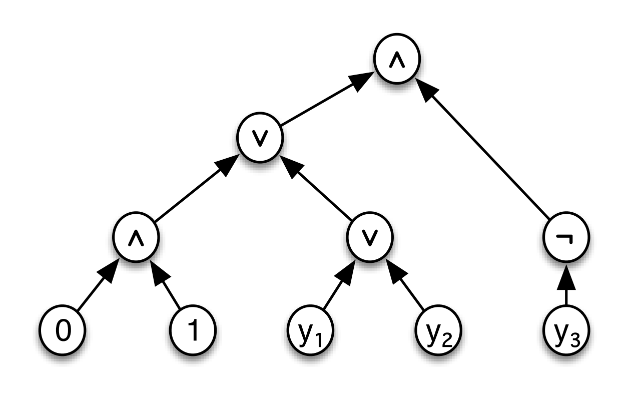
Physical circuits and boolean combinatorial circuits

- A physical circuit consists of gates that perform logical AND, OR and NOT.
- We will model such a circuit by a boolean combinatorial circuit which is a labelled DAG with
  - **Source nodes**: these are the inputs of the circuit and may be hardwired to 0 or 1, or labelled with some variable.
  - **Intermediate** nodes: these correspond to the gates of the circuit and are labelled with $\wedge$ (AND), $\vee$ (OR) or $\neg$ (NOT).
    - $\wedge$, $\vee$ gates have two incoming and one outgoing edge
    - $\neg$ gates have one incoming and one outgoing edge
  - **Sink node**: corresponds to the output of the circuit and has no outgoing edges.

Example circuit



A circuit C with 2 hardwired source nodes, 3 variable inputs y_1, y_2, y_3 and 5 logical gates.

Circuit-SAT: a first $\mathcal{NP}$-complete problem

Evaluating a circuit:

- edges are wires that carry the value of their tail node;
- intermediate nodes perform their label operation on their incoming edges, pass the result along their outgoing edge;
- the value of the circuit is the value of its output node.

Definition 12 (`Circuit-SAT`).

Given a circuit C, is there an assignment of truth values to its inputs that causes the output to evaluate to 1?

It is easy to see that `Circuit-SAT` is in $\mathcal{NP}$. Cook and Levin showed that it is $\mathcal{NP}$-complete.

SAT is $\mathcal{NP}$-complete

Lemma 13.

`Circuit-SAT` $\leq_P$ `SAT`

Intuitively, this reduction should not be too difficult: formulas and circuits are just different ways of representing boolean functions and translating from one to the other should be easy.

The following two boolean connectives are very useful.

1. $(\phi_1 \Rightarrow \phi_2)$ is a shorthand for $(\overline{\phi_1} \vee \phi_2)$.
   Intuition: if $\phi_1 = 1$, then $\phi_2 = 1$ too (o.w., $(\phi_1 \Rightarrow \phi_2) = 0$).
2. $(\phi_1 \Leftrightarrow \phi_2)$ is a shorthand for $((\phi_1 \Rightarrow \phi_2) \wedge (\phi_2 \Rightarrow \phi_1))$, which may be expanded to $(\overline{\phi_1} \vee \phi_2) \wedge (\phi_1 \vee \overline{\phi_2})$.
   This clause evaluates to 1 if and only if $\phi_1 = \phi_2$.

Transforming a circuit C into a formula ϕ

Consider an arbitrary instance of `Circuit-SAT`, that is, a circuit C with source nodes, intermediate nodes and an output node.

For **every** node v in C, we introduce to ϕ

- a variable x_v that encodes the truth value computed by node v in C;
- clauses that ensure that x_v takes on the same value as the output of node v given its inputs

Then any satisfying truth assignment for the circuit C will imply that ϕ is satisfiable, while, if ϕ is satisfiable, setting the variable inputs of C to the truth values of their corresponding variables in ϕ will result in C computing an output with value 1.

ϕ is the conjunction of the following clauses

1. If v is a source node corresponding to a variable input of the circuit C, we do not add any clause.
2. If v is a source node hardwired to 0, add $(\overline{x_v})$.
3. If v is a source node hardwired to 1, add (x_v).
4. If v is the output node, add (x_v).
5. If v is a node labelled by NOT and its input edge is from node u, add $(x_v \Leftrightarrow \overline{x_u})$.
6. If v is a node labelled by OR and its input edges are from nodes u and w, add $(x_v \Leftrightarrow (x_u \vee x_w))$.
7. If v is a node labelled by AND and its input edges are from nodes u and w, add $(x_v \Leftrightarrow (x_u \wedge x_w))$.

Transforming C into ϕ requires polynomial time

This completes our construction of the clauses of ϕ.

For example, for the circuit in slide 34, we construct the following formula.

$$\begin{aligned}\phi \quad = \quad & (\neg x_1) \wedge (x_2) \wedge (x_6 \Leftrightarrow (x_1 \wedge x_2)) \wedge (x_7 \Leftrightarrow (x_3 \vee x_4)) \wedge \\ & (x_8 \Leftrightarrow \neg x_5) \wedge (x_9 \Leftrightarrow (x_6 \vee x_7)) \wedge (x_{10} \Leftrightarrow (x_9 \wedge x_8)) \wedge (x_{10})\end{aligned}$$

The construction is polynomial in the size of the input circuit *(why?)*.

Moreover, every clause consists of at most three literals, once ϕ is in CNF (*exercise*).

Proof of equivalence

- $\Rightarrow$ Let T_C be a truth assignment to the variable inputs of C that causes C to evaluate to 1. Propagate T_C to assign a truth value to every node v in C. Define a truth assignment T_ϕ for ϕ as follows: x_v takes on the truth value of v, for every node v in C. Then T_ϕ satisfies ϕ.
- $\Leftarrow$ Suppose ϕ has a satisfying truth assignment. Then the truth values of the variables of ϕ that correspond to inputs in C satisfy C: the clauses in ϕ guarantee that, for every node in C, the value assigned to that node is exactly what that node computes in C. Since $\phi = 1$, C evaluates to 1.

Independent set

So far, we have stated (with or without proofs) that

- `Circuit-SAT` is $\mathcal{NP}$-complete
- `Circuit-SAT` $\leq_P$ `SAT`
- `SAT` $\leq_P$ `3SAT`

$\Rightarrow$ `SAT` and `3SAT` are $\mathcal{NP}$-complete.

Is `IS(D)` *as "hard" as* `SAT`*?*

Independent set

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$\Rightarrow$ `SAT` and `3SAT` are $\mathcal{NP}$-complete.

Claim 1.

`IS(D)` *is* $\mathcal{NP}$*-complete.*

Proof.

Reduction from `3SAT`. $\square$

Structure of the proof

Given an arbitrary instance formula ϕ of `3SAT`, we need to transform it into a graph G and an integer k, so that

1. The transformation is completed in polynomial time.
2. The instance (G, k) is a **yes** instance of `IS(D)`

   if and only if

   ϕ is a **yes** instance of `3SAT`.

Structure of the proof

Given an arbitrary instance formula ϕ of `3SAT`, we need to transform it into a graph G and an integer k, so that

1. The transformation is completed in polynomial time.
2. G has an independent set of size at least k

   if and only if

   ϕ is satisfiable

Example: given

$$\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$$

construct

$$(G, k)$$

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Remark 1.

- *Heart of reduction* $\texttt{X} \leq_P \texttt{Y}$*: understand why some small instance of* Y *makes it difficult.*
- *For* `IS(D)`*, such an instance is a triangle: it's not clear which of its vertices to add to our independent set.*

Gadgets!

When reducing from `3SAT`, we often use gadgets. Gadgets are constructions that ensure:

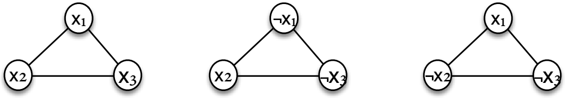
1. Consistency of truth values in a truth assignment: once x_i is assigned a truth value, we must henceforth consistently use it under this truth value.
2. Clause constraints: since ϕ is in CNF, we must provide a way to satisfy every clause. Equivalently, we must exhibit at least one literal that is set to 1 in every clause.

In effect, these gadgets will allow us to derive a valid and satisfying truth assignment for ϕ when the transformed instance is a **yes** instance of our problem, so we can prove equivalence of the two instances.

Gadgets for IS(D)

Clause constraint gadget: for every clause, introduce a triangle where a node is labelled by a literal in the clause.

Example: $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$



- Hence our graph G consists of m isolated triangles.
- The max independent set in this graph has size m: pick one vertex from every triangle. So we will set $k = m$.

Goal: derive a truth assignment from our independent set S.
Idea: when a node from a triangle is added to S, set the corresponding literal to 1.

Consistency gadgets

2. Is this truth assignment consistent?
    - Suppose x_1 was picked from the first triangle.
    - Can still pick $\overline{x_1}$ from the second triangle!
    - But then we are setting x_1 to both 1 and 0.

    $\Rightarrow$ This is obviously **not** a valid truth assignment!

Consistency of truth assignment: must ensure that we cannot add a node labelled x_i **and** a node labelled $\overline{x_i}$ to our independent set.

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Consistency of truth assignment: must ensure that we cannot add a node labelled x_i **and** a node labelled $\overline{x_i}$ to our independent set.

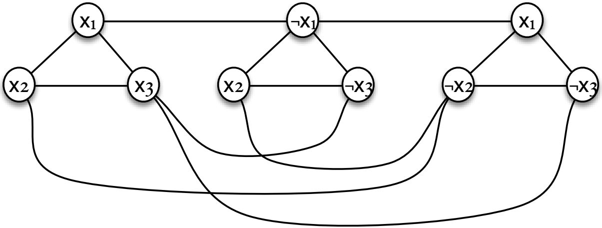
Consistency gadget: add edges between all occurrences of x_i and $\overline{x_i}$, for every i, in G.

Constructed instance (G, k) of IS(D)

Example: given the formula ϕ below (n=m=3)

$$\phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor \neg x_3) \land (x_1 \lor \neg x_2 \lor \neg x_3),$$

the derived graph G is as follows:



Set k=m=3; the input instance R(ϕ) to IS(D) is (G, 3).

Remark: the construction requires time polynomial in the size of ϕ.

Proof of equivalence

We need to show that

ϕ is satisfiable

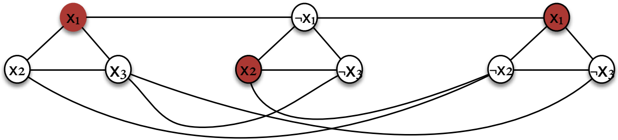
if and only if

G has an independent set of size at least m

Proof of equivalence, reverse direction

- Suppose that G has an independent set S of size m.
- Then **every** triangle contributes one node to S.
- Define the following truth assignment
  - Set the literal corresponding to that node to 1.
  - Any variables left unset by this assignment may be set to 0 or 1 arbitrarily.

$$\phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor \neg x_3) \land (x_1 \lor \neg x_2 \lor \neg x_3)$$



Independent set S = $\{x_1, x_2, x_1\}$

Derived truth assignment: $x_1=1$, $x_2=1$, $x_3=0$

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1. is valid
2. satisfies ϕ

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We need to show that this truth assignment

1. is valid: by construction, $x_i, \overline{x_i}$ cannot **both** appear in S.
2. satisfies ϕ: since **every** triangle contributes one node to S, every clause has a true literal, thus every clause is satisfied.

Proof of equivalence, forward direction

- Now suppose there is a satisfying truth assignment for ϕ.
- Then there is (at least) one `True` literal in every clause.
- Construct an independent set S as follows:
  From every triangle, add to S a node labelled by such a literal; hence S has size m.

We claim that S thus constructed is indeed an independent set.

Proof of equivalence, forward direction

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- Then there is (at least) one `True` literal in every clause.
- Construct an independent set S as follows:
  From every triangle, add to S a node labelled by such a literal; hence S has size m.

We claim that S thus constructed is indeed an independent set.

1. S would not be an independent set *if* there was an edge between any two nodes in it.
2. Since all nodes in S belong to *different* triangles, an edge implies that the two nodes are labelled by opposite literals.
3. Impossible: S only contains `True` literals (so it cannot contain both a literal and its negation).