

Analysis of Algorithms, CSOR W4231

CSOR W4231

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Data compression and Huffman coding

Outline

- 1 Data compression
- 2 Symbol codes and optimal lossless compression
- 3 Prefix codes
- 4 Prefix codes and trees
- 5 The Huffman algorithm

Today

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Data compression: find compact representations of data

Data compression **standards**

- ▶ jpeg for image transmission
- ▶ mp3 for audio content, mpeg2 for video transmission
- ▶ utilities: gzip, bzip2

All of the above use the **Huffman algorithm** as a basic building block.

Data representation

- ▶ An organism's genome consists of *chromosomes* (giant linear DNA molecules)
- ▶ *Chromosome maps*: sequences of hundreds of millions of bases (symbols from $\{A, C, G, T\}$).
- ▶ **Goal:** store a chromosome map with 200 million bases.

How do we represent a chromosome map?

Data representation

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- ▶ **Goal**: store a chromosome map with 200 million bases.

How do we represent a chromosome map?

- ▶ **Encode** every symbol that appears in the sequence **separately** by a **fixed length binary string**.
- ▶ **Codeword** $c(x)$ for symbol x : a binary string encoding x of length $\ell(x)$

Example code

- ▶ Alphabet $\mathcal{A} = \{A, C, G, T\}$ with 4 symbols
- ▶ Encode each symbol with 2 bits

alphabet symbol x	codeword $c(x)$
A	00
C	01
G	10
T	11

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- ▶ Input sequence: $ACGTAA$

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Output of encoding: the concatenation of the codewords for every symbol in the input sequence

Example

- ▶ Input sequence: $ACGTAA$
- ▶ Output: $c(A)c(C)c(G)c(T)c(A)c(A) = 000110110000$
- ▶ Total length of encoding = $6 \cdot 2 = 12$ bits.

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Symbol codes

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Examples of symbol codes

- ▶ $C_0 = \{00, 01, 10, 11\}$ is a symbol code for $\{A, C, G, T\}$.
- ▶ ASCII encoding system: every character and special symbol on the computer keyboard is encoded by a different 7-bit binary string.

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Remark 1.

C_0 and ASCII are *fixed-length* symbol codes: each codeword has the same length.

Unique decodability

Decoding C_0 ?

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- ▶ read 2 bits of the output;
- ▶ print the symbol corresponding to this codeword;
- ▶ continue with the next 2 bits.

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- ▶ C_0 , ASCII: *distinct input sequences have distinct encodings*

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Definition 1.

A symbol code is **uniquely decodable** if, for any two **distinct** input sequences, their encodings are distinct.

Lossless compression

- ▶ **Lossless compression:** compress and decompress without errors.
- ▶ Uniquely decodable codes allow for **lossless compression**.
- ▶ A symbol code achieves **optimal lossless compression** when it produces an encoding of **minimum length** for its input (among all uniquely decodable symbol codes).
- ▶ Huffman algorithm: provides a symbol code that achieves **optimal** lossless compression.

Fixed-length vs variable-length codes

Chromosome map consists of 200 million bases as follows:

alphabet symbol x	frequency $freq(x)$
A	110 million
C	5 million
G	25 million
T	60 million

Fixed-length vs variable-length codes

Chromosome map consists of 200 million bases as follows:

alphabet symbol x	frequency $freq(x)$
A	110 million
C	5 million
G	25 million
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- ▶ A appears much more often than the other symbols.
- ⇒ It might be best to encode A with fewer bits.
- ▶ Unlikely that the **fixed-length encoding** C_0 is optimal

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Variable-length encodings

Code C_1

alphabet symbol x	codeword $c(x)$
A	0
C	00
G	10
T	1

Variable-length encodings

Code C_1

alphabet symbol x	codeword $c(x)$
A	0
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G	10
T	1

- ▶ C_1 is **not** unique decodable! E.g., 101110: *how to decode it?*

Variable-length encodings

Code C_2

alphabet symbol x	codeword $c(x)$
A	0
C	110
G	111
T	10

Variable-length encodings

Code C_2

alphabet symbol x	codeword $c(x)$
A	0
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- ▶ C_2 is uniquely decodable.
- ▶ C_2 is such that no codeword is a **prefix** of another.

Variable-length encodings

Code C_2

alphabet symbol x	codeword $c(x)$
A	0
C	110
G	111
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- ▶ C_2 is uniquely decodable.
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Definition 2 (prefix codes).

A symbol code is a **prefix code** if no codeword is a prefix of another.

Decoding prefix codes

1. Scan the binary string from left to right until you've seen enough bits to match a codeword;
2. Output the symbol corresponding to this codeword.
 - ▶ Since no other codeword is a prefix of this codeword or contains it as a prefix, this sequence of bits cannot be used to encode any other symbol.
3. Continue starting from the next bit of the bit string.

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Thus prefix codes allow for

- ▶ **unique decoding**;
- ▶ **fast decoding** (the end of a codeword is instantly recognizable).

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- ▶ **fast decoding** (the end of a codeword is instantly recognizable).

Examples of prefix codes: C_0 , C_2

Prefix codes and optimal lossless compression

- ▶ Decoding a prefix code is very fast.
- ⇒ Would like to focus on prefix codes (rather than **all** uniquely decodable symbol codes) for achieving **optimal lossless compression**.
- ▶ Information theory guarantees this: for every uniquely decodeable code, exists a prefix code with the **same codeword lengths**
- ▶ So we can solely focus on prefix codes for optimal compression.

Compression gains from variable-length prefix codes

Chromosome map: *do we gain anything by using C_2 instead of C_0 when compressing the map of 200 million bases?*

Input	
symbol x	$freq(x)$
A	110 million
C	5 million
G	25 million
T	60 million

Code C_0	
x	$c(x)$
A	00
C	01
G	10
T	11

Code C_2	
x	$c(x)$
A	0
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G	111
T	10

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Input		Code C_0		Code C_2	
symbol x	$freq(x)$	x	$c(x)$	x	$c(x)$
A	110 million	A	00	A	0
C	5 million	C	01	C	110
G	25 million	G	10	G	111
T	60 million	T	11	T	10

- ▶ C_0 : 2 bits \times 200 million symbols = 400 million bits
- ▶ C_2 : $1 \cdot 110 + 3 \cdot 5 + 3 \cdot 25 + 2 \cdot 60 = 320$ million bits
- ▶ Improvement of 20% in this example

The optimal prefix code problem

Input:

- ▶ Alphabet $\mathcal{A} = \{a_1, \dots, a_n\}$
- ▶ Set $P = \{p_1, \dots, p_n\}$ of empirical probabilities over \mathcal{A} such that $p_i = \Pr[a_i]$

Output: a binary prefix code $C^* = \{c(a_1), c(a_2), \dots, c(a_n)\}$ for (\mathcal{A}, P) , where codeword $c(a_i)$ has length ℓ_i and is such that its expected length

$$L(C^*) = \sum_{a_i \in \mathcal{A}} p_i \cdot \ell_i$$

is **minimum** among all binary prefix codes.

Example

Chromosome example

Input	
symbol x	$\Pr(x)$
A	110/200
C	5/200
G	25/200
T	60/200

Code C_0	
x	$c(x)$
A	00
C	01
G	10
T	11

Code C_2	
x	$c(x)$
A	0
C	110
G	111
T	10

- ▶ $L(C_0) = 2$
- ▶ $L(C_2) = 1.6$
- ▶ *Coming up:* C_2 is the output of the Huffman algorithm, hence an optimal encoding for (\mathcal{A}, P) .

Today

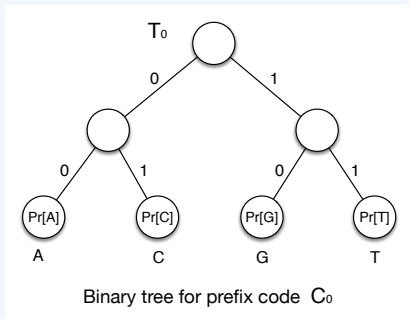
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Prefix codes and trees

- ▶ A **binary tree** T is a rooted tree such that each node that is not a leaf has at most two children.
- ▶ Binary tree for a prefix code: a branch to the left represents a 0 in the encoding and a branch to the right a 1.

Code C_0

x	$c(x)$
A	00
C	01
G	10
T	11

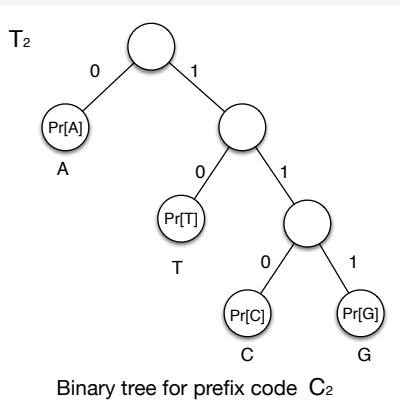


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Code C_2

x	$c(x)$
A	0
C	110
G	111
T	10



Properties of binary trees representing prefix codes

1. *Where do alphabet symbols appear in the tree?*
2. *What do codewords correspond to in the tree?*
3. *Consider the tree corresponding to the optimal prefix code. Can it have internal nodes with one child?*

Properties of binary trees representing prefix codes

1. Symbols must appear at the leaves of the tree T (*why?*)
 $\Rightarrow T$ has n leaves.
2. Codewords $c(a_i)$ are given by **root-to-leaf paths**.

Recall that ℓ_i is the length of the codeword $c(a_i)$ for input symbol a_i . Therefore, on the tree T , ℓ_i corresponds to the depth of a_i (we assume that the root is at depth 0).

\Rightarrow Can rewrite the **expected length** of the prefix code as:

$$L(C) = \sum_{a_i \in \mathcal{A}} p_i \cdot \ell_i = \sum_{1 \leq i \leq n} p_i \cdot \text{depth}_T(a_i) = L(T).$$

3. **Optimal tree must be full**: all internal nodes must have exactly two children (*why?*).

Claim 1.

There is an optimal prefix code, with corresponding tree T^ , in which the two lowest frequency characters are assigned to leaves that are siblings in T^* at maximum depth.*

Claim 1.

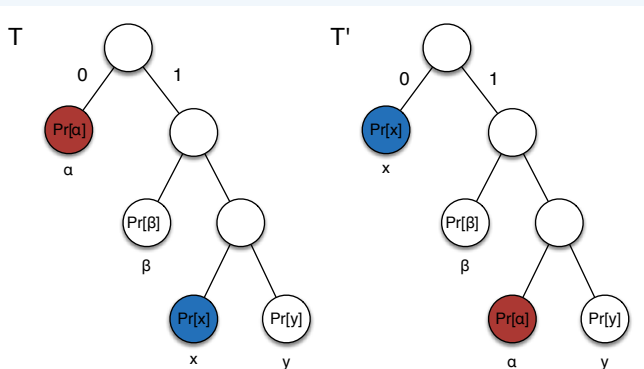
There is an optimal prefix code, with corresponding tree T^ , in which the two lowest frequency characters are assigned to leaves that are siblings in T^* at maximum depth.*

Proof.

By an **exchange argument**: start with a tree for an optimal prefix code and **transform** it into T^* . □

Proof of Claim 1

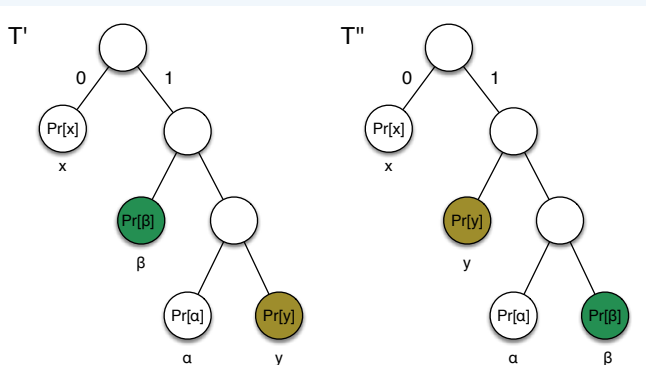
- ▶ Let T be the tree for the optimal prefix code.
- ▶ Let α, β be the two symbols with the smallest probabilities, that is, $\Pr[\alpha] \leq \Pr[\beta] \leq \Pr[s]$ for all $s \in \mathcal{A} - \{\alpha, \beta\}$.
- ▶ Let x and y be the two siblings at maximum depth in T .



We want $L(T) \geq L(T')$

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- ▶ Let x and y be the two siblings at maximum depth in T .



We want $L(T') \geq L(T'')$

How do the expected lengths of the two trees compare?

$$\begin{aligned}L(T) - L(T') &= \sum_{a_i \in \mathcal{A}} \Pr[a_i] \cdot \text{depth}_T(a_i) - \sum_{a_i \in \mathcal{A}} \Pr[a_i] \cdot \text{depth}_{T'}(a_i) \\&= \Pr[\alpha] \cdot \text{depth}_T(\alpha) + \Pr[x] \cdot \text{depth}_T(x) \\&\quad - \Pr[\alpha] \cdot \text{depth}_{T'}(\alpha) - \Pr[x] \cdot \text{depth}_{T'}(x) \\&= \Pr[\alpha] \cdot \text{depth}_T(\alpha) + \Pr[x] \cdot \text{depth}_T(x) \\&\quad - \Pr[\alpha] \cdot \text{depth}_T(x) - \Pr[x] \cdot \text{depth}_T(\alpha) \\&= (\Pr[\alpha] - \Pr[x]) \cdot (\text{depth}_T(\alpha) - \text{depth}_T(x)) \geq 0\end{aligned}$$

- ▶ The third equality follows from the exchange.
- ▶ Similarly, exchanging β and y in T' yields $L(T') - L(T'') \geq 0$.
- ▶ Hence $L(T) - L(T'') \geq 0$.
- ▶ Since T is optimal, it must be $L(T) = L(T'')$.
- ▶ So T'' is also optimal.

The claim follows by setting T^* to be T'' .

Building the optimal tree

Claim 1 tells us how to build the optimal tree **greedily!**

1. Find the two symbols with the lowest probabilities.
2. Remove them from the alphabet and replace them with a new **meta-character** with probability equal to the sum of their probabilities.
 - ▶ **Idea:** this meta-character will be the parent of the two deleted symbols in the tree.
3. Recursively construct the optimal tree using this process.

Greedy algorithms: *make a local (myopic) decision at every step that optimizes some criterion and eventually show that this is the optimal way for building the entire solution.*

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Huffman algorithm

Huffman(\mathcal{A}, P)

if $|\mathcal{A}| = 2$ **then**

 Encode one symbol using 0 and the other using 1

end if

Let α and β be the two symbols with the lowest probabilities

Let ν be a new meta-character with probability $\Pr[\alpha] + \Pr[\beta]$

Let $\mathcal{A}_1 = \mathcal{A} - \{\alpha, \beta\} + \{\nu\}$

Let P_1 be the new set of probabilities over \mathcal{A}_1

$T_1 = \text{Huffman}(\mathcal{A}_1, P_1)$

return T as follows: replace leaf node ν in T_1 by an internal node, and add two children labelled α and β below ν .

Remark 3.

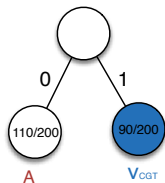
Output of Huffman procedure is a binary tree T ; the code for (\mathcal{A}, P) is its corresponding prefix code.

Example: recursive Huffman for chromosome map

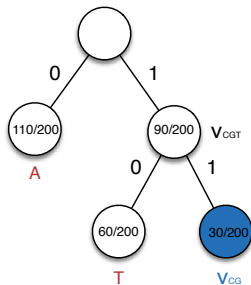
Recursive call 1: $\text{Huffman}(\{A, C, G, T\}, \{\frac{110}{200}, \frac{5}{200}, \frac{25}{200}, \frac{60}{200}\})$

Recursive call 2: $\text{Huffman}(\{A, \nu_{CG}, T\}, \{\frac{110}{200}, \frac{30}{200}, \frac{60}{200}\})$

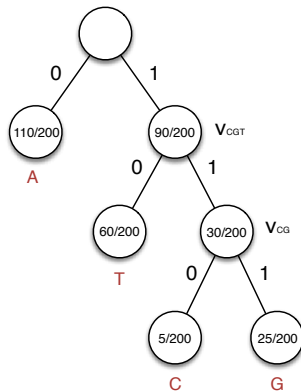
Recursive call 3: $\text{Huffman}(\{A, \nu_{CGT}\}, \{\frac{110}{200}, \frac{90}{200}\})$



End of rec. call 3



End of rec. call 2



End of rec. call 1

Proof: by induction on the size of the alphabet $n \geq 2$.

- ▶ **Base case.** For $n = 2$, Huffman is optimal.
- ▶ **Hypothesis.** Assume that Huffman returns the optimal prefix code for alphabets of n symbols.
- ▶ **Induction Step.** Let \mathcal{A} be an alphabet of size $n + 1$, P the corresponding set of probabilities.

Let T_1 be the optimal (by the hypothesis) tree returned by our algorithm for (\mathcal{A}_1, P_1) , where \mathcal{A}_1, P_1, T_1 as in the pseudocode. Let T be the final tree returned for (\mathcal{A}, P) by our algorithm. We claim that T is optimal.

We will prove the claim by contradiction. **Assume** T^* is the optimal tree for (\mathcal{A}, P) such that

$$L(T^*) < L(T). \tag{1}$$

Fact 3.

Let T be a binary tree representing a prefix code. If we replace sibling leaves α, β in T by a meta-character ν where $\Pr[\nu] = \Pr[\alpha] + \Pr[\beta]$, we obtain a tree T_1 such that

$$L(T) = L(T_1) + (\Pr[\alpha] + \Pr[\beta]).$$

Proof.

Notation: $d_T(a_i) = \text{depth}_T(a_i)$

- α, β are sibling leaves in T , hence $d_T(\alpha) = d_T(\beta)$.
- T differs from T_1 only in that α, β are replaced by ν . Since $d_{T_1}(\nu) = d_T(\alpha) - 1$, we obtain

$$\begin{aligned} L(T) - L(T_1) &= \Pr[\alpha]d_T(\alpha) + \Pr[\beta]d_T(\beta) - (\Pr[\alpha] + \Pr[\beta])d_{T_1}(\nu) \\ &= \Pr[\alpha] + \Pr[\beta]. \end{aligned} \tag{2}$$



Correctness (cont'd)

- ▶ Claim 1 guarantees there is such an optimal tree for (\mathcal{A}, P) where α, β appear as siblings at maximum depth.
- ▶ W.l.o.g. assume that T^* is such an optimal tree. By Fact 3, if we replace siblings α, β in T^* by ν' where $\Pr[\nu'] = \Pr[\alpha] + \Pr[\beta]$, the resulting tree T_1^* satisfies $L(T^*) = L(T_1^*) + (\Pr[\alpha] + \Pr[\beta])$.
- ▶ Similarly, the tree T returned by the Huffman algorithm satisfies $L(T) = L(T_1) + (\Pr[\alpha] + \Pr[\beta])$.
- ▶ By the induction hypothesis, we have $L(T_1^*) \geq L(T_1)$ since T_1 is optimal for alphabets of size n . Hence

$$L(T^*) = L(T_1^*) + \Pr[\alpha] + \Pr[\beta] \geq L(T_1) + \Pr[\alpha] + \Pr[\beta] = L(T), \quad (3)$$

where the inequality follows from the induction hypothesis.

- ▶ Equation (3) contradicts Assumption (1). Thus T must be optimal.

Implementation and running time

1. Straightforward implementation: $O(n^2)$ time
2. Store the alphabet symbols in a min priority queue implemented as a **binary min-heap** with **keys** their probabilities
 - ▶ **Operations:** Initialize ($O(n)$), Extract-min ($O(\log n)$), Insert ($O(\log n)$)Total time: $O(n \log n)$ time

For an iterative implementation of Huffman, see your textbook.

Example: iterative Huffman for chromosome map

Input (\mathcal{A}, P)

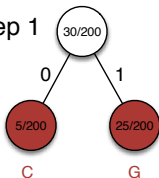
symbol x	$\text{Pr}(x)$
A	110/200
C	5/200
G	25/200
T	60/200

→

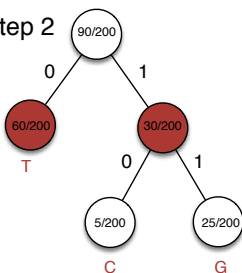
Output code

symbol x	$c(x)$
A	0
C	110
G	111
T	10

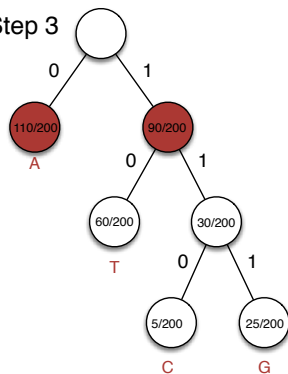
Step 1



Step 2



Step 3



Beyond Huffman coding

- ▶ Huffman algorithm provides an optimal **symbol** code.
- ▶ Codes that encode larger blocks of input symbols might achieve better compression.
- ▶ Storage on noisy media: *what if a bit of the output of the compressor is flipped?*
 - ▶ Decompression cannot carry through.
 - ▶ Need **error correction** on top of compression.