Analysis of Algorithms, I
CSOR W4231.002

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Outline

1. Recap
2. Matrix chain multiplication
3. A first attempt: brute-force
4. A second attempt: divide and conquer
5. A Dynamic Programming (DP) solution
6. Organizing DP computations

Today

Review of last lecture

Greedy algorithms: cache maintenance

- The offline problem
- An optimal algorithm for the offline problem: Farthest-in-Future (FF)
- Proof of optimality of FF
- The online problem

Today

Matrix chain multiplication example

Example 1.

Input: matrices A_1, A_2, A_3 of dimensions 6×1, 1×5, 5×2

Output:

- a way to compute the product $A_1 A_2 A_3$ so that the number of arithmetic operations performed is minimized;
- the minimum number of arithmetic operations required.

Useful observations

Remark 1.

- *We do not want to compute the actual product.*
- *Matrix multiplication is associative but not commutative (in general). Hence a solution to our problem corresponds to a parenthesization of the product.*
- *We want the optimal parenthesization and its cost, that is, the parenthesization that minimizes the number of arithmetic operations, as well as that number.*

Estimating #arithmetic operations

- Let A, B be matrices of dimensions $m \times n$, $n \times p$.
- Let $C = AB$. Then C is an $m \times p$ matrix such that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}.$$

$\Rightarrow$ c_{ij} requires n scalar multiplications, $n - 1$ additions

$\Rightarrow$ #arithmetic operations to compute c_{ij} is dominated by #scalar multiplications

- Total #scalar multiplications to fill in C is mnp

Minimizing #scalar multiplications for $A_1A_2A_3$

Input: A_1, A_2, A_3 of dimensions 6×1, 1×5, 5×2 respectively

Recall that, given a parenthesization of the input matrices, its cost is the total # scalar multiplications to compute the product.
Two ways of computing $A_1A_2A_3$:

1. $(A_1A_2)A_3$: first compute A_1A_2, then multiply it by A_3
   - $6 \cdot 1 \cdot 5$ scalar multiplications for A_1A_2
   - $6 \cdot 5 \cdot 2$ scalar multiplications for $(A_1A_2)A_3$

   $\Rightarrow$ 90 scalar multiplications in total

2. $A_1(A_2A_3)$: first compute A_2A_3, then multiply A_1 by A_2A_3
   - $1 \cdot 5 \cdot 2$ scalar multiplications for A_2A_3
   - $6 \cdot 1 \cdot 2$ scalar multiplications for $A_1(A_2A_3)$

   $\Rightarrow$ 22 scalar multiplications in total

Remark 2.

Solution $A_1(A_2A_3)$ improves over $(A_1A_2)A_3$ by over 75%.

(Fully) Parenthesized products of matrices

Definition 2.

A product of matrices is fully parenthesized if it is

1. a single matrix; or
2. the product of two fully parenthesized matrices, surrounded by parentheses.

Examples: $((A_1 A_2) A_3)$ and $(A_1 (A_2 A_3))$ are fully parenthesized.

Remark: we will henceforth refer to a *full parenthesization* simply as a *parenthesization*.

Matrix chain multiplication

Input: n matrices $A_1, A_2, \ldots, A_n$, with dimensions $p_{i-1} \times p_i$, for $1 \leq i \leq n$.

Output:

1. the **optimal** parenthesization of the input (that is, the one incurring the minimum cost);
2. its cost.

Example: the optimal parenthesization for Example 1 is $(A_1(A_2A_3))$ and its cost is 22.

Remark 3.

- *We might want the optimal solution and its cost, or just the cost.*
- *The optimal solution might not be unique; of course, the optimal cost is unique.*

Today

Brute-force approach

- $A_1, \ldots, A_n$ are matrices of dimensions $p_{i-1} \times p_i$ for $1 \leq i \leq n$.
- Consider the product $A_1 \cdots A_n$.
- Let $P(n) = \#$parenthesizations of the product $A_1 \cdots A_n$.
- Then $P(0) = 0$, $P(1) = 1$, $P(2) = 1$
- For $n > 2$, by Definition 2, for every possible parenthesization, there is a $1 \leq k \leq n-1$ such that the parenthesized product looks like

$$((A_1 A_2 \cdots A_k)(A_{k+1} \cdots A_n))$$

Computing #possible parenthsizations

- Given k, the #parenthesizations for the product

$$((A_1 A_2 \cdots A_k)(A_{k+1} \cdots A_n))$$

can be computed recursively:

$$P(k) \cdot P(n-k)$$

- There are $n-1$ possible values for k. Hence

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k), \text{ for } n > 1$$

Bounding $P(n)$

- We may obtain a crude yet sufficient for our purposes **lower bound** for $P(n)$ as follows

$$\begin{aligned} P(n) &\geq P(1) \cdot P(n-1) + P(2) \cdot P(n-2) \\ &\geq P(n-1) + P(n-2) \end{aligned} \tag{1}$$

- By strong induction on n, we can show that $P(n) \geq F_n$, the n-th Fibonacci number.

- By Problem 6a in Homework 1, $P(n) = \Omega(2^{n/2})$.
  - In fact, $P(n) = \Omega(2^{2n}/n^{3/2})$ (e.g., see your textbook).

$\Rightarrow$ Brute force requires exponential time.

Today

A second attempt: divide and conquer

Notation:

1. $(A_i \cdots A_j)$ is a parenthesization of the product $A_i \cdots A_j$.
2. $A_{1,n}$ is the optimal parenthesization of the product $A_1 \cdots A_n$, that is, the one that incurs the minimum cost.

- Consider a parenthesization for $A_1 \cdots A_n$. By Definition 2, it is the product of two fully parenthesized subproducts; hence for some $1 \leq k \leq n-1$

$$(A_1 \cdots A_n) = ((A_1 \cdots A_k)(A_{k+1} \cdots A_n))$$

- In particular, there exists $1 \leq k^* \leq n-1$ such that

$$A_{1,n} = (A_1 \cdots A_{k^*})(A_{k^*+1} \cdots A_n)$$

Optimal substructure

Notation: $A_{i,j}$ is the optimal parenthesization of the product $A_i \cdots A_j$.

Fact 3.

There exists k^* *such that* $1 \leq k^* \leq n-1$ *and*

$$A_{1,n} = A_{1,k^*}\ A_{k^*+1,n}.$$

Hence the optimal parenthesization of the input can be decomposed into the optimal parenthesizations of two subproblems.

The cost of multiplying two matrices

- Recall that matrix A_i has dimensions $p_{i-1} \times p_i$. Hence
  - $(A_1 \cdots A_k)$ is a $p_0 \times p_k$ matrix,
  - $(A_{k+1} \cdots A_n)$ is a $p_k \times p_n$ matrix.
- The #scalar multiplications required for multiplying matrix $(A_1 \cdots A_k)$ by matrix $(A_{k+1} \cdots A_n)$ is

$$p_0 p_k p_n.$$

Proof of optimal substructure

Notation: $A_{i,j}$ is the optimal parenthesization of $A_i \cdots A_j$.

- By Definition 2, exists k^* such that

$$A_{1,n} = ((A_1 \cdots A_{k^*})(A_{k^*+1} \cdots A_n))$$

  Then the cost of $A_{1,n}$ is the **sum** of

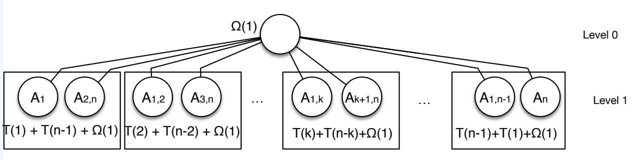
  1. the costs of the subproblems $A_1 \cdots A_{k^*}$, $A_{k^*+1} \cdots A_n$;
  2. the fixed cost $p_0 p_{k^*} p_n$ of multiplying $(A_1 \cdots A_{k^*})$ by $(A_{k^*+1} \cdots A_n)$.

- *If a solution to a subproblem was not optimal, replacing it by a better one in the overall solution would yield a cheaper overall solution, thus contradicting optimality of* $A_{1,n}$.

$\Rightarrow$ Hence $(A_1 \cdots A_{k^*}), (A_{k^*+1} \cdots A_n)$ must be **optimal** parenthesizations themselves.

Recursive computation of $A_{1,n}$

- **Idea:** compute the cost of the optimal parenthesization recursively.
- **Issue:** we do not know k^*!
- **Solution:** consider every possible value of k.
  - So we must solve $n-1$ *large* subproblems, one for every $1 \leq k \leq n-1$; each *large* subproblem involves solving two subproblems, that is, $A_{1,k}$, $A_{k+1,n}$, and combining them.



Exponential-time recursion

Notation: $T(n)$ = **time** required to optimally parenthesize a product of n matrices.

- At level 0, there are $n-1$ *large* subproblems. Finding the one that incurs the minimum cost requires time $\Omega(1)$.
- The k-th *large* subproblem at level 1 requires time $T(k) + T(n-k) + \Omega(1)$.
- Note that $T(1) \geq 1$, $T(2) \geq 2$.
- Therefore,

$$T(n) \quad \geq \quad 1 + \sum_{k=1}^{n-1} \Big(T(k) + T(n-k) + 1 \Big).$$

- Then $T(n) \geq T(n-1) + T(n-2)$.
- Hence $T(n) \geq F_n$ (see our argument for $P(n)$).

$\Rightarrow$ The recursive algorithm requires $\Omega(2^{n/2})$ time.

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Are we really that far from an efficient solution?

Recall Fibonacci problem from HW1: exponential recursive algorithm, polynomial iterative solution

How?

1. Overlapping subproblems: spectacular redundancy in computations of recursion tree
2. Easy-to-compute recurrence for combining the smaller subproblems: $F_n = F_{n-1} + F_{n-2}$
3. Small number of subproblems: only solved $n - 1$ subproblems.
4. Iterative, bottom-up computations: we computed the subproblems from smallest (F_0, F_1) to largest (F_n), iteratively.

Elements of DP in matrix chain multplication

Our problem exhibits similar properties.

1. We showed overlapping subproblems.
2. We have implicitly formulated a recurrence for the cost of the optimal parenthesization in terms of the costs of the optimal parenthesizations of appropriate subproblems. We will show that the recurrence can be computed in polynomial time, given solutions to subproblems.
3. We will show a polynomial number of subproblems..
4. We will solve the subproblems in a bottom-up fashion, from smallest to largest.

The cost of multiplying two matrices

- Recall that A_i is a $p_{i-1} \times p_i$ matrix. Hence
  - $(A_i \cdots A_k)$ is a $p_{i-1} \times p_k$ matrix,
  - $(A_{k+1} \cdots A_j)$ is a $p_k \times p_j$ matrix.
- Then the #scalar multiplications required for computing the product $(A_i \cdots A_k)(A_{k+1} \cdots A_j)$ is

$$p_{i-1} p_k p_j.$$

Introducing subproblems: a first attempt

For $1 \leq j \leq n$, define

$$OPT(1, j) = \text{optimal cost for computing } A_1 \cdots A_j$$

$$OPT(1, j) = \begin{cases} 0 & \text{, if } j = 1 \\ \min\limits_{1 \leq k < j} \left\{ OPT(1, k) + OPT(k+1, j) + p_0 p_k p_j \right\} & \text{, if } j > 1 \end{cases}$$

△ Does not work: the subproblems are not both of the same form as the original problem *(why?)*.

Introducing more subproblems

For $1 \le i \le j \le n$, define

$$OPT(i,j) = \text{optimal cost for computing } A_i \cdots A_j$$

$$OPT(i,j) = \begin{cases} 0 & \text{, if } i = j \\ \min\limits_{i \le k < j} \left\{ OPT(i,k) + OPT(k+1,j) + p_{i-1}p_kp_j \right\} & \text{, if } i < j \end{cases}$$

Remark 4.

- *Only* $\Theta(n^2)$ *subproblems.*
- *If subproblems are computed from smaller to larger, then only* $\Theta(j-i) = \Theta(n)$ *work per subproblem: each term inside the* min *computation requires time* $O(1)$ *(why?).*

Today

Bottom-up computation of subproblems

Define matrix $M[1:n, 1:n]$, $S[1:n-1, 2:n]$ such that

$$\begin{aligned} M[i,j] &= OPT(i,j), && \text{for } 1 \leq i \leq j \leq n \\ S[i,j] &= k, \text{ if } A_{i,j} = A_{i,k}A_{k+1,j}, && \text{for } 1 \leq i < j \leq n \end{aligned}$$

- Only need fill in the upper triangle of M, where $i \leq j$
- Start from the main diagonal, proceed diagonal by diagonal
- Last entry to fill in: $M[1,n]$, the cost of the optimal parenthesization of the entire product $A_1 \cdots A_n$
- Running time: $O(n^3)$
  - $\Theta(n^2)$ entries to fill in
  - each entry requires $\Theta(j-i) = O(n)$ work
- Space: $\Theta(n^2)$

Example

Input

- 6×1 matrix A_1
- 1×5 matrix A_2
- 5×2 matrix A_3
- 2×3 matrix A_4

Output

- the cost of the optimal parenthesization of $A_1 A_2 A_3 A_4$ (by filling in the dynamic programming table M)

Computing the cost of the optimal parenthesization in $O(n^3)$ (from CLRS)

```
MATRIX-CHAIN-ORDER(p)
n = p.length - 1
let m[1..n, 1..n] and s[1..n-1, 2..n] be new tables
for i = 1 to n
    m[i,i] = 0
for l = 2 to n                    // l is the chain length
    for i = 1 to n - l + 1
        j = i + l - 1
        m[i,j] = ∞
        for k = i to j - 1
            q = m[i,k] + m[k+1,j] + p_{i-1} p_k p_j
            if q < m[i,j]
                m[i,j] = q
                s[i,j] = k
return m and s
```

Reconstructing the optimal parenthesization (from CLRS)

```
PRINT-OPTIMAL-PARENS(s, i, j)
if i == j
    print "A"_i
else print "("
    PRINT-OPTIMAL-PARENS(s, i, s[i, j])
    PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
    print ")"
```

Memoized recursion

Use the original recursive algorithm together with M:

- initialize M to ∞ above the main diagonal and to 0 on the main diagonal.
- to solve a subproblem, look up its value in M
  - if it is ∞, solve the subproblem **and** store its cost in M;
  - else, directly use its value from M.

Remark 5.

- *The memoized recursive algorithm solves every subproblem once, thus overcoming the main source of inefficiency of the original recursive algorithm.*
- *Running time:* $O(n^3)$.

Memoized recursion pseudocode (from CLRS)

```
MEMOIZED-MATRIX-CHAIN(p)
n = p.length - 1
let m[1..n, 1..n] be a new table
for i = 1 to n
    for j = i to n
        m[i, j] = ∞
return LOOKUP-CHAIN(m, p, 1, n)

LOOKUP-CHAIN(m, p, i, j)
if m[i, j] < ∞
    return m[i, j]
if i == j
    m[i, j] = 0
else for k = i to j - 1
        q = LOOKUP-CHAIN(m, p, i, k)
            + LOOKUP-CHAIN(m, p, k + 1, j) + p_{i-1} p_k p_j
        if q < m[i, j]
            m[i, j] = q
return m[i, j]
```

Dynamic programming vs Divide & Conquer

- They both combine solutions to subproblems to generate a solution to the whole problem.
- However, divide and conquer starts with a large problem and divides it into small pieces.
- While dynamic programming works from the bottom up, solving the smallest subproblems first and building optimal solutions to steadily larger problems.