Analysis of Algorithms, I

CSOR W4231.002

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Outline

1. Review of last lecture
   - $\texttt{IS(D)} \leq_P \texttt{3SAT}$
2. Representative $\mathcal{NP}$-complete problems
3. Minimum-weight Set Cover

Today

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   - $\texttt{IS(D)} \leq_P \texttt{3SAT}$

2. Representative $\mathcal{NP}$-complete problems

3. Minimum-weight Set Cover

Complexity classes $\mathcal{P}$, $\mathcal{NP}$ and $\mathcal{NP}$-complete

Definition 1.

We define $\mathcal{P}$ to be the set of problems that can be solved by polynomial-time algorithms.

Definition 2.

We define $\mathcal{NP}$ to be the set of decision problems that have an efficient certifier.

Fact 3.

$\mathcal{P} \subseteq \mathcal{NP}$

Definition 4.

A problem $X(D)$ is $\mathcal{NP}$-complete if

1. $X(D) \in \mathcal{NP}$ and
2. for all $Y \in \mathcal{NP}$, $Y \leq_P X$.

Why we should care whether a problem is $\mathcal{NP}$-complete

If a problem is $\mathcal{NP}$-complete, we need to *stop looking for efficient algorithms for the general problem.*

Instead we have a number of options, such as

1. approximation algorithms
   - mathematically rigorous basis to study heuristics
   - distinguish between various optimization problems in terms of how well they can be approximated
2. work on interesting special cases
3. study the average performance of the algorithm
4. *heuristics*

How do we show that a problem is $\mathcal{NP}$-complete?

Suppose we had an $\mathcal{NP}$-complete problem X.

To show that another problem Y is $\mathcal{NP}$-complete, we use transitivity of reductions. So we "only" need show that

1. $Y \in \mathcal{NP}$
2. $X \leq_P Y$

The first $\mathcal{NP}$-complete problem

Theorem 5 (Cook-Levin).

Circuit SAT is $\mathcal{NP}$-complete.

Satisfiability of boolean functions

`SAT`: Given a formula ϕ in CNF with n variables and m clauses, is ϕ satisfiable?

`3SAT`: Given a formula ϕ in CNF with n variables and m clauses such that each clause has exactly 3 literals, is ϕ satisfiable?

`Circuit-SAT`: Given a boolean combinatorial circuit C, is there an assignment of truth values to its inputs that causes the output to evaluate to 1?

Lemma 6.

Circuit-SAT $\leq_P$ *SAT*, *SAT* $\leq_P$ *3SAT and 3SAT* $\leq_P$ $IS(D)$

Independent set

So far, we have stated (with or without proofs) that

- `Circuit-SAT` is $\mathcal{NP}$-complete
- `Circuit-SAT` $\leq_P$ `SAT`
- `SAT` $\leq_P$ `3SAT`

$\Rightarrow$ `SAT` and `3SAT` are $\mathcal{NP}$-complete.

Is `IS(D)` *as "hard" as* `SAT`*?*

Independent set

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- `Circuit-SAT` $\leq_P$ `SAT`
- `SAT` $\leq_P$ `3SAT`

$\Rightarrow$ `SAT` and `3SAT` are $\mathcal{NP}$-complete.

Claim 1.

`IS(D)` *is* $\mathcal{NP}$*-complete.*

Proof.

Reduction from `3SAT`. □

Structure of the proof

Given an arbitrary instance formula ϕ of `3SAT`, we need to transform it into a graph G and an integer k, so that

1. The transformation is completed in polynomial time.
2. The instance (G, k) is a **yes** instance of `IS(D)`

   if and only if

   ϕ is a **yes** instance of `3SAT`.

Structure of the proof

Given an arbitrary instance formula ϕ of `3SAT`, we need to transform it into a graph G and an integer k, so that

1. The transformation is completed in polynomial time.
2. G has an independent set of size at least k

   if and only if

   ϕ is satisfiable

Example: given

$$\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$$

construct

$$(G, k)$$

Structure of the proof

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Remark 1.

- *Heart of reduction* `X` $\leq_P$ `Y`*: understand why some small instance of Y makes it difficult.*
- *For* `IS(D)`*, such an instance is a triangle: it's not clear which of its vertices to add to our independent set.*

Gadgets!

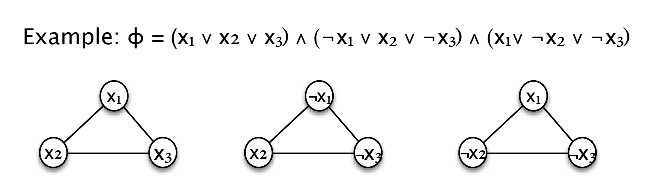
When reducing from `3SAT`, we often use gadgets. Gadgets are constructions that ensure:

1. Consistency of truth values in a truth assignment: once x_i is assigned a truth value, we must henceforth consistently use it under this truth value.
2. Clause constraints: since ϕ is in CNF, we must provide a way to satisfy every clause. Equivalently, we must exhibit at least one literal that is set to 1 in every clause.

In effect, these gadgets will allow us to derive a valid and satisfying truth assignment for ϕ when the transformed instance is a **yes** instance of our problem, so we can prove equivalence of the two instances.

Gadgets for IS(D)

Clause constraint gadget: for every clause, introduce a triangle where a node is labelled by a literal in the clause.



- Hence our graph G consists of m isolated triangles.
- The max independent set in this graph has size m: pick one vertex from every triangle. So we will set $k = m$.

Goal: derive a truth assignment from our independent set S.
Idea: when a node from a triangle is added to S, set the corresponding literal to 1.

Consistency gadgets

2. Is this truth assignment consistent?
   - Suppose x_1 was picked from the first triangle.
   - Can still pick $\overline{x_1}$ from the second triangle!
   - But then we are setting x_1 to both 1 and 0.

   $\Rightarrow$ This is obviously **not** a valid truth assignment!

Consistency of truth assignment: must ensure that we cannot add a node labelled x_i **and** a node labelled $\overline{x_i}$ to our independent set.

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Consistency of truth assignment: must ensure that we cannot add a node labelled x_i **and** a node labelled $\overline{x_i}$ to our independent set.

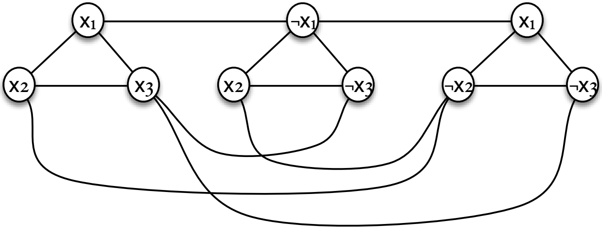
Consistency gadget: add edges between all occurrences of x_i and $\overline{x_i}$, for every i, in G.

Constructed instance (G, k) of IS(D)

Example: given the formula ϕ below (n=m=3)

$$\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3),$$

the derived graph G is as follows:



Set k=m=3; the input instance R(ϕ) to IS(D) is (G, 3).

Remark: the construction requires time polynomial in the size of ϕ.

Proof of equivalence

We need to show that

ϕ is satisfiable

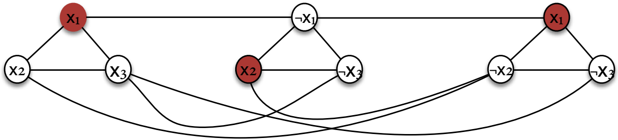
if and only if

G has an independent set of size at least m

Proof of equivalence, reverse direction

- Suppose that G has an independent set S of size m.
- Then **every** triangle contributes one node to S.
- Define the following truth assignment
  - Set the literal corresponding to that node to 1.
  - Any variables left unset by this assignment may be set to 0 or 1 arbitrarily.

$\phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor \neg x_3) \land (x_1 \lor \neg x_2 \lor \neg x_3)$



Independent set S = {x_1, x_2, x_1}

Derived truth assignment: $x_1=1$, $x_2=1$, $x_3=0$

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We need to show that this truth assignment

1. is valid
2. satisfies ϕ

Proof of equivalence, reverse direction

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- Define the following truth assignment
  - Set the literal corresponding to that node to 1.
  - Any variables left unset by this assignment may be set to 0 or 1 arbitrarily.

We need to show that this truth assignment

1. is valid: by construction, $x_i, \overline{x_i}$ cannot **both** appear in S.
2. satisfies ϕ: since **every** triangle contributes one node to S, every clause has a true literal, thus every clause is satisfied.

Proof of equivalence, forward direction

- Now suppose there is a satisfying truth assignment for ϕ.
- Then there is (at least) one true literal in every clause.
- Construct an independent set S as follows:
  From every triangle, add to S a node labelled by such a literal; hence S has size m.

We claim that S thus constructed is indeed an independent set.

Proof of equivalence, forward direction

- Now suppose there is a satisfying truth assignment for ϕ.
- Then there is (at least) one true literal in every clause.
- Construct an independent set S as follows:
  From every triangle, add to S a node labelled by such a literal; hence S has size m.

We claim that S thus constructed is indeed an independent set.

1. S would not be an independent set *if* there was an edge between any two nodes in it.
2. Since all nodes in S belong to *different* triangles, an edge implies that the two nodes are labelled by opposite literals.
3. Impossible: *all* literals in S evaluate to 1.

Common pitfalls when showing $\mathcal{NP}$-completeness

1. Carry out the reduction in the wrong direction
2. Reduce from a problem not known to be $\mathcal{NP}$-complete
3. Exponential-time transformations
   - Subsets, permutations
4. Neglect to carefully prove both directions of equivalence of the original and the derived instances; that is, x is a **yes** instance of X *if and only if* $y = R(x)$ is a **yes** instance of Y
5. Neglect to show that the problem is in $\mathcal{NP}$

Suggestions

- You should think carefully which problem is most suitable to reduce from
- In absence of other ideas, reduce from 3SAT

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The Traveling Salesman Problem (TSP)

Tour: a *simple* cycle that visits *every* vertex exactly once.

Definition 7 (TSP(D)).

Given n cities $\{1, \ldots, n\}$, a set of non-negative distances d_{ij} between every pair of cities and a budget B, is there a tour of length $\leq B$?

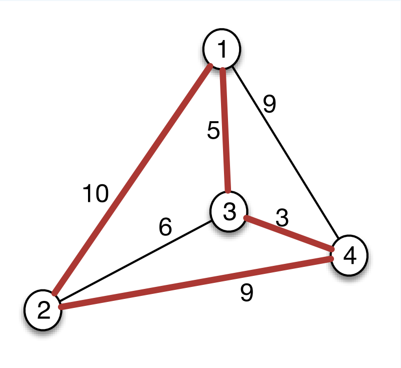
Equivalently, is there a permutation π such that

1. $\pi(1) = \pi(n+1) = 1;$ that is, we start and end at city 1
2. the total distance travelled satisfies

$$\sum_{i=1}^{n} d_{\pi(i)\pi(i+1)} \leq B$$

Application: Google street view car

Example instance of TSP



Depending on the distances, TSP instances may be

- *Asymmetric*: $d_{ij} \neq d_{ji}$
- *Symmetric*: $d_{ij} = d_{ji}$
- *Metric*: satisfy the triangle inequality $d_{ij} \leq d_{ik} + d_{kj}$
- *Euclidean*: e.g., cities are in $\mathcal{R}^2$ hence city i corresponds to point (x_i, y_i); then $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

A related problem and hardness of `TSP(D)`

`Hamiltonian Cycle`: Given a graph $G = (V, E)$, is there a simple cycle that visits every vertex exactly once?

Claim 2.

`Hamiltonian Cycle` *is* $\mathcal{NP}$*-complete.*

Proof: Reduction from `3SAT` (e.g., see your textbook).

Claim 3.

`TSP(D)` *is* $\mathcal{NP}$*-complete.*

Proof: reduction from `Hamiltonian Cycle`.

Proof of Claim 3 (`Hamiltonian Cycle` $\leq_P$ `TSP(D)`)

1. Start from an arbitrary instance of `Hamiltonian Cycle`, that is, an undirected graph $G = (V, E)$.
2. Construct the following instance $(G' = (V', E', w), B)$ of `TSP(D)`: G' is a *complete* weighted graph with $V' = V$ such that for every edge $e \in E'$,

$$w_e = \begin{cases} 1, & \text{if } e \in E \\ 2, & \text{otherwise} \end{cases}$$

3. Set the budget $B = n$.

This completes the reduction transformation.

Equivalence of the instances is straightforward:

- If G has a hamiltonian cycle, that cycle is a tour of length n in G'.
- If G' has a tour of length n, it must consist of edges of weight 1 *(why?)*; thus all these edges appear in G.

Concluding remarks on TSP

- Claim 2 also holds for directed Hamiltonian cycle. An exact analog of the proof of Claim 3 then shows that asymmetric TSP is $\mathcal{NP}$-complete.
- It is possible to reduce `Hamiltonian cycle` to `Euclidean TSP`, thus showing that even `Euclidean TSP` is $\mathcal{NP}$-complete.
- However, these problems are not similar in terms of how well they can be approximated: it is possible to provide very good approximate solutions to `Euclidean TSP`, which is not the case for `Symmetric TSP`.

Packing and partitioning problems

- Set Packing: given a set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and a number k, is there a collection of at least k subsets such that no two of them intersect?

- 3D-Matching: Given disjoint sets B, G, H, each of size n, and a set of triples $T \subseteq B \times G \times H$, is there a set of n triples in T, no two of which have an element in common?

  Reduction from 3SAT.

Numerical problems

- Subset sum: Given natural numbers $w_1, \ldots, w_n$ and a (large) target weight W, is there a subset of $w_1, \ldots, w_n$ that adds up exactly to W?

  Applications: cryptography, scheduling

- Minimum-weight solution to linear equations: Given a system of linear equations in n variables with integer constants, and an integer $B \leq n$, does it have a rational solution with at most B non-zero entries?

  Applications: coding theory, signal processing

Similar problems with very different complexities

$\mathcal{NP}$	$\mathcal{P}$
max cut	min cut
longest path	shortest path
3D matching	matching
Hamiltonian cycle	Euler cycle
3-colorability	2-colorability
3-SAT	2-SAT
LCS of n sequences	LCS of 2 sequences

More on $\mathcal{NP}$-completeness:

- *Computers and Intractability: A guide to the theory of $\mathcal{NP}$-completeness*, by Garey and Johnson
- *Computational Complexity*, by C. Papadimitriou

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Minimum-weight Set Cover

Input

- a set $E = \{e_1, e_2, \ldots, e_n\}$ of n elements
- a collection of subsets of these elements $S_1, S_2, \ldots, S_m$, where each $S_j \subseteq E$
- a non-negative weight w_j for every subset S_j

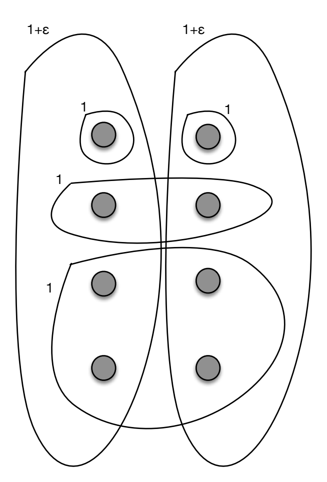
Output

A minimum-weight collection of subsets that cover all of E.

In symbols: find an $I \subseteq \{1, \ldots, m\}$ such that $\cup_{i \in I} S_i = E$ and $\sum_{i \in I} w_i$ is minimum.

(`Unweighted Set Cover`: $w_j = 1$ for all j)

Example instance of `Set Cover`



$n = 8$ ground elements, $m = 6$ subsets with weights
$w_1 = w_2 = w_3 = w_4 = 1$, $w_5 = w_6 = 1 + \epsilon \quad (0 < \epsilon < 1/2)$

Motivation: detect computer viruses

Goal: detect features of viruses that do not occur in typical applications

- Ground elements: computer viruses ($n \approx 150$)
- Sets: labelled by some three-byte sequence occurring in these viruses but not occurring in typical computer applications ($m \approx 21000$); each set consisted of all the viruses that contained the three-byte sequence
- Goal: output a small number of such sequences (much smaller than 150) that *cover* all known viruses

Reduction via generalization

Claim 4.

`Set-Cover(D)` *is* $\mathcal{NP}$*-complete.*

Proof.

Reduction from `VC(D)`.

- Let $E = \{e_1, \ldots, e_m\}$ be the set of edges in the graph
  - These are the ground elements we are trying to *cover*.
- Let S_j be the set of edges (ground elements) that are *covered* by vertex i.
  - A vertex j *covers* all edges adjacent to it.
- Set $w_j = 1$ for all $1 \leq j \leq n$.

Equivalence of instances: input graph has a vertex cover of size k if and only if E can be covered by k sets. □